deriving a DEQ from rate information

A 30 year old woman accepts an engineering position with a starting salary of \$ 30,000 per year. Her salary $S(t)$ increases exponentially, with $S(t) = 30\, e^{t/20}$ thousand dollars after t years.

Meanwhile 12% of her salary is deposited continuously in a retirement account, which accumulates interest at a continuous annual rate of 6%.

a) Estimate ΔA in terms of Δt to derive the DEQ satisfied by the amount $A(t)$ in her retirement account after t years.

b) Compute $A(40)$, the amount available for her retirement at age 70.

not part of problem:

context info (K\$):

} $S(40) = 30\, e^{2} \approx 222$ retirement salary value; total salary: $\int_0^{40} 30 e^{.05t}\,dt \approx 3833$

} $A(0) = 0$: no money in retirement when start.

Think of Δt as a paycheck period (typically 2 weeks $= \frac{2}{52}$ yr)

$$\Delta A(t) = .12\ S(t)\ \Delta t + A(t)\ (.06\,\Delta t)$$

- K\$/yr → $S(t)$: annual salary; Δt: fraction of year } units of yr!
- $.12\,S(t)$: paycheck amount; $.12\,S(t)\,\Delta t$: retirement fund deposit that paycheck
- $A(t)$: current amount; $.06$: annual interest rate; $.06\,\Delta t$: interest rate per paycheck
- $A(t)(.06\,\Delta t)$: interest on amount for one paycheck period

"continuously" depositing is obviously an idealization, but it gives us a simple model (interest rates are not constant etc so none of the numbers can be trusted in the real world)

however, it allows us to take the limit $\Delta t \to 0$ to get a DEQ rather than just some discrete equation:

$$\frac{\Delta A(t)}{\Delta t} = \frac{.12\,(30 e^{t/20})\,\Delta t + .06\,A(t)\,\Delta t}{\Delta t} = .12(30)e^{\frac{t}{20}} + .06\,A(t)$$

$\downarrow \lim_{\Delta t \to 0}$

linear DEQ: $\boxed{\dfrac{dA}{dt} = .12(30)e^{\frac{t}{20}} + .06\,A}$ → standard form $\left[\dfrac{dA}{dt} - .06A = .12(30)\,e^{\frac{t}{20}}\right]$ $\quad \frac{1}{20} = .05$

$e^{\int -.06\,dt} = e^{-.06t}$ integrating factor

$$\frac{d}{dt}(A e^{-.06t}) = .12(30)\, e^{+.05t} e^{-.06t} = .12(30)\, e^{-.01t}$$

$$A e^{-.06t} = \int .12(30)\, e^{-.01t}\,dt = \frac{.12(30)}{-.01} e^{-.01t} + C$$

$$A = -\frac{.12(30)}{.01} e^{.05t} + C e^{.06t} \qquad \text{I.C.: } 0 = A(0) = -\frac{.12(30)}{.01} + C \to C = \frac{.12(30)}{.01}$$

IVP soln: $A = \underbrace{\frac{.12(30)}{.01}}_{360} (e^{.06t} - e^{.05t}) \longrightarrow A(40) = 360\,(e^{2.4} - e^{2}) \approx 1308.283301$ K\$

$\approx \$1{,}308{,}283.30 \approx \$1{,}308{,}000 \approx \boxed{\$1{,}310{,}000}$

none of these numbers can be taken too seriously so too many significant figures don't make sense

Followup: How to factor in inflation? Real dollar value?