resonance calculation (2) (back to example)

$$y''+4y'+5y = B_0 \cos\omega t$$

you don't need to remember any formulas apart from the amplitude relation:

$c_1\cos\omega t + c_2 \sin\omega t$

$\hookrightarrow (c_1, c_2) \longrightarrow A = \sqrt{c_1^2+c_2^2}$

$$\begin{aligned}
5\,[y_p &= c_3\cos\omega t + c_4\sin\omega t\,]\\
4\,[y_p' &= -\omega c_3 \sin\omega t + \omega c_4 \cos\omega t]\\
1\,[y_p'' &= -\omega^2 c_3\cos\omega t - \omega^2 c_4 \sin\omega t]
\end{aligned}$$

$$\begin{aligned}
y_p''+4y_p'+5y_p &= [(5-\omega^2)c_3 + 4\omega c_4]\cos\omega t\\
&\quad + [-4\omega c_3 + (5-\omega^2)c_4]\sin\omega t\\
&= B_0\cos\omega t
\end{aligned}$$

$$\begin{aligned}
(5-\omega^2)c_3 + 4\omega c_4 &= B_0\\
-4\omega c_3 + (5-\omega^2)c_4 &= 0
\end{aligned}$$

$$\begin{bmatrix} 5-\omega^2 & 4\omega \\ -4\omega & 5-\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{(5-\omega^2)^2+16\omega^2}\begin{bmatrix} 5-\omega^2 & -4\omega \\ 4\omega & 5-\omega^2 \end{bmatrix}\begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$= \frac{B_0}{(5-\omega^2)^2+16\omega^2}\begin{bmatrix} 5-\omega^2 \\ 4\omega \end{bmatrix}$$

steady state soln:

$$y_p = \frac{B_0}{(5-\omega^2)^2+16\omega^2}\left[(5-\omega^2)\cos\omega t + 4\omega\sin\omega t\right]$$

amplitude response function:

$$A(\omega) = \frac{B_0\sqrt{(5-\omega^2)^2+16\omega^2}}{(5-\omega^2)^2+16\omega^2} = \frac{B_0}{[(5-\omega^2)^2+16\omega^2]^{1/2}}$$

$$= B_0[\ldots]^{-1/2}$$

peak:

$$0 = A'(\omega) = -\tfrac{1}{2}\left[(5-\omega^2)^2+16\omega^2\right]^{-3/2}\underbrace{\left[2(5-\omega^2)(-2\omega)+36\omega\right]}_{4\omega(\underbrace{-5+\omega^2+9}_{=0})}$$

$\omega^2 = -4$ no real solution, no peak

only slightly underdamped: $Q = \frac{\sqrt{5}}{4} \approx 0.56 \geq 0.5$,

need $Q > \frac{1}{\sqrt{2}} \approx 0.707$ for peak to appear,

no resonance, biggest response at zero frequency