

resonance calculation

$$y'' + 4y' + 5y = 10 \cos 3t$$

$k_0 = 4$ $\omega_0 = \sqrt{5} \approx 2.24$ $\omega = 3 > \omega_0$
 $\zeta = 1/4$ $T = 2\pi/\omega_0 \approx 2.81$ $\leftrightarrow r = \pm 3i$
 $Q \approx 2.24/4 \approx 0.56 > 1/2$ (no root overlap)
barely underdamped
 $r^2 + 4r + 5 = 0 \rightarrow r = -2 \pm i$
 $e^{rt} = e^{-2t} e^{\pm it} = e^{-2t} (\cos t \pm i \sin t)$
 $y_h = e^{-2t} (c_1 \cos \omega_0 t + c_2 \sin \omega_0 t)$
↑ dies away by $5t = \frac{\pi}{2}$
"transient"

$y_p = \text{most gen soln of } (D^2 + \omega^2) y = 0$
 $\omega = 3$
 ω_{general}

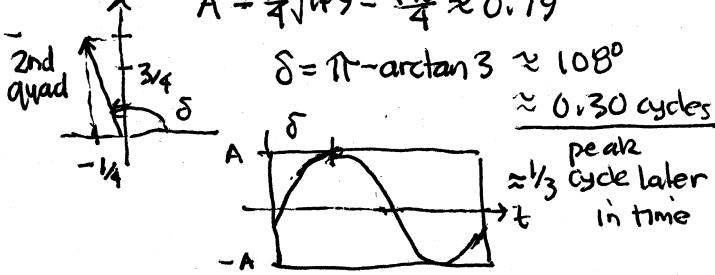
5 [$y_p = c_3 \cos 3t + c_4 \sin 3t$]
[$y_p' = -3c_3 \sin 3t + 3c_4 \cos 3t$]
1 [$y_p'' = -9c_3 \cos 3t - 9c_4 \sin 3t$]

$$y_p'' + 4y_p' + 5y_p = [(5-9)c_3 + 12c_4] \cos 3t = 10 \cos 3t + [-12c_3 + (5-9)c_4] \sin 3t$$

$$\begin{bmatrix} -4 & 12 \\ -12 & -4 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -4 & -12 \\ 12 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix}$$

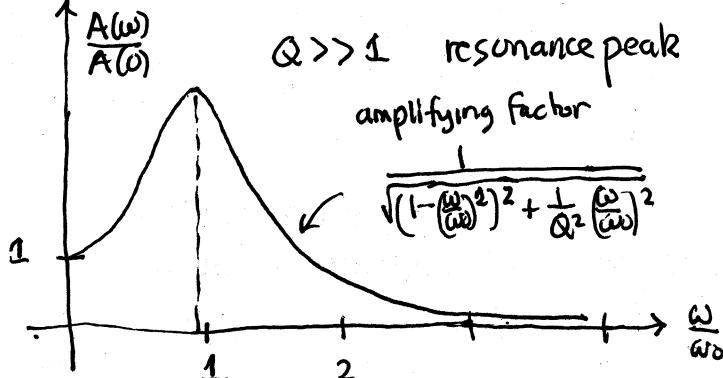
$16 + 144 = 160$

$$y_p = \frac{1}{4} (-\cos 3t + 3 \sin 3t) = A \cos(3t - \delta) \leftarrow \text{"steady state soln"}$$



$$y = y_h + y_p = e^{-2t} (c_1 \cos \omega_0 t + c_2 \sin \omega_0 t) + \frac{1}{4} (-\cos 3t + 3 \sin 3t)$$

$$\frac{\omega}{\omega_0} = \frac{3}{\sqrt{5}} \approx 1.34$$



$$y'' + k_0 y' + \omega_0^2 y = B_0 \cos \omega t$$

$$r^2 + k_0 r + \omega_0^2 = 0$$

$$r = -k \pm i \omega_0 \text{ or } R_1, R_2 < 0$$

$$y_h = \dots \text{ transient. dies away after } 5t$$

$\omega_0^2 [y_p = C_3 \cos \omega t + C_4 \sin \omega t]$
 $R_0 [y_p' = -\omega_0 C_3 \sin \omega t + \omega_0 C_4 \cos \omega t]$
1 [$y_p'' = -\omega_0^2 C_3 \cos \omega t - \omega_0^2 C_4 \sin \omega t$]

$$y_p'' + k_0 y_p' + \omega_0^2 y_p = \begin{bmatrix} (R_0^2 - \omega^2) C_3 + k_0 \omega C_4 \\ -k_0 \omega C_3 + (R_0^2 - \omega^2) C_4 \end{bmatrix} \cos \omega t + \begin{bmatrix} -k_0 \omega C_3 + (R_0^2 - \omega^2) C_4 \\ -k_0 \omega C_3 + (R_0^2 - \omega^2) C_4 \end{bmatrix} \sin \omega t$$

$$\begin{bmatrix} (R_0^2 - \omega^2) & k_0 \omega \\ -k_0 \omega & R_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

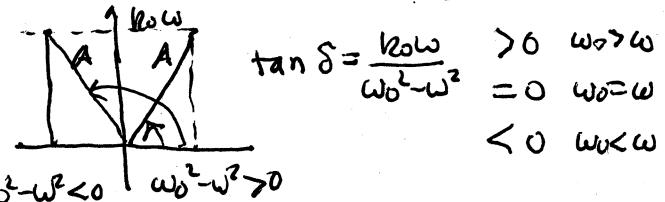
$$\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \frac{\begin{bmatrix} k_0 \omega^2 - \omega^2 & -k_0 \omega \\ R_0 \omega & R_0^2 - \omega^2 \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ 0 \end{bmatrix}}{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2}$$

$$= \frac{B_0}{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2} \begin{bmatrix} \omega_0^2 - \omega^2 \\ k_0 \omega \end{bmatrix}$$

$$y_p = \frac{B_0}{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2} ((\omega_0^2 - \omega^2) \cos \omega t + k_0 \omega \sin \omega t)$$

$$y = y_h + y_p$$

transient steady state soln



$$A(A) = \frac{B_0 \sqrt{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2}}{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2} = \frac{B_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2}}$$

$$= \frac{B_0 / \omega_0^2}{\sqrt{(-\frac{\omega^2}{\omega_0^2})^2 + \frac{k_0^2 \omega^2}{\omega_0^2}}} = \frac{(B_0 / \omega_0^2) \leq A(0)}{\sqrt{(-\frac{\omega^2}{\omega_0^2})^2 + \frac{k_0^2 \omega^2}{\omega_0^2}}}$$

resonance calculation (2) (back to example)

$$y'' + 4y' + 5y = B_0 \cos \omega t$$

$$S[y_p = c_3 \cos \omega t + c_4 \sin \omega t]$$

$$\uparrow [y_p' = -\omega c_3 \sin \omega t + \omega c_4 \cos \omega t]$$

$$I[y_p'' = -\omega^2 c_3 \cos \omega t - \omega^2 c_4 \sin \omega t]$$

$$\begin{aligned} y_p'' + 4y_p' + 5y_p &= [(5-\omega^2)c_3 + 4\omega c_4] \cos \omega t \\ &\quad + [-4\omega c_3 + (5-\omega^2)c_4] \sin \omega t \\ &= B_0 \cos \omega t \end{aligned}$$

$$(5-\omega^2)c_3 + 4\omega c_4 = B_0$$

$$-4\omega c_3 + (5-\omega^2)c_4 = 0$$

$$\begin{bmatrix} 5-\omega^2 & 4\omega \\ -4\omega & 5-\omega^2 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{(5-\omega^2)^2 + 16\omega^2} \begin{bmatrix} 5-\omega^2 & -4\omega \\ 4\omega & 5-\omega^2 \end{bmatrix} \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$= \frac{B_0}{(5-\omega^2)^2 + 16\omega^2} \begin{bmatrix} 5-\omega^2 \\ 4\omega \end{bmatrix}$$

steady state soln:

$$y_p = \frac{B_0}{(5-\omega^2)^2 + 16\omega^2} [(5-\omega^2) \cos \omega t + 4\omega \sin \omega t]$$

amplitude response function:

$$A(\omega) = \frac{B_0 \sqrt{(5-\omega^2)^2 + 16\omega^2}}{(5-\omega^2)^2 + 16\omega^2} = \frac{B_0}{[(5-\omega^2)^2 + 16\omega^2]^{1/2}}$$

$$= B_0 [\dots]^{-1/2}$$

peak:

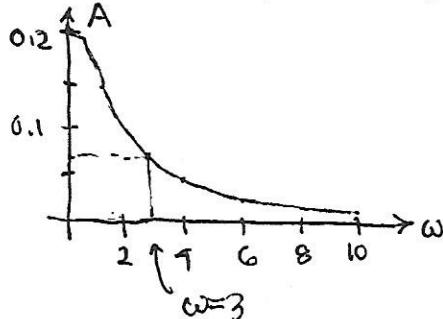
$$0 = A'(\omega) = -\frac{1}{2} [(5-\omega^2)^2 + 16\omega^2]^{-3/2} \underbrace{[2(5-\omega^2)(-2\omega) + 32\omega]}_{4\omega(-5+\omega^2+8)} = 0$$

$\omega^2 = -3$ no real solution,
no peak

only slightly underdamped: $Q = \frac{\sqrt{5}}{4} \approx 0.56 \gtrsim 0.5$,

need $Q > \frac{1}{\sqrt{2}} \approx 0.707$ for peak to appear,

no resonance, biggest response at zero frequency



you don't need to remember any formulas apart from the amplitude relation:

$$c_1 \cos \omega t + c_2 \sin \omega t$$

$$\hookrightarrow (c_1, c_2) \rightarrow A = \sqrt{c_1^2 + c_2^2}$$