exercise: coupled damped harmonic oscillators with "compatible" damping SOLUTION (2)
REDUCTION OF ORDER

$\vec{x}' = a\vec{x} + \vec{F} \rightarrow$ set $\vec{F} = 0$ for this calculation

$$a = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{bmatrix} \rightarrow (\lambda_i) = \begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ -\frac{1}{2} + i\frac{\sqrt{11}}{2} \\ -\frac{1}{2} - i\frac{\sqrt{11}}{2} \end{bmatrix} \quad B = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & \frac{1}{6} + \frac{i}{6}\sqrt{11} & \frac{1}{6} - \frac{i\sqrt{11}}{6} \\ -\frac{1}{2} - \frac{i}{2}\sqrt{3} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{6} - \frac{i}{6}\sqrt{11} & -\frac{1}{6} + \frac{i\sqrt{11}}{6} \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

divide each column by 2nd entry
to compare with 2x2 eigenvector DE soln

$$B_{new} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{1}{2} - \frac{i}{2}\sqrt{11} & \frac{1}{2} + \frac{i\sqrt{11}}{2} \\ -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i}{2}\sqrt{11} & -\frac{1}{2} - \frac{i\sqrt{11}}{2} \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_1 & \vec{b}_2 & \vec{b}_2 \\ \lambda_+\vec{b}_1 & \bar{\lambda}_+\vec{b}_1 & \lambda_-\vec{b}_2 & \bar{\lambda}_-\vec{b}_2 \end{bmatrix}$$

$$\underbrace{}_{\vec{\mathcal{b}}_1} \quad \underbrace{}_{\vec{\mathcal{b}}_2 = \bar{\vec{\mathcal{b}}}_1} \quad \underbrace{}_{\vec{\mathcal{b}}_3} \quad \underbrace{}_{\vec{\mathcal{b}}_4 = \bar{\vec{\mathcal{b}}}_3}$$

lower 2 component vector in 4 component eigenvectors comes from derivative of $e^{\lambda_i t}$!

$$\chi = \begin{bmatrix} \vec{X} \\ \vec{X}' \end{bmatrix} = B_{new}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \sum_{i=1}^{4} y_i \vec{\mathcal{b}}_i = \sum_{i=1}^{4} c_i e^{\lambda_i t}\vec{\mathcal{b}}_i = \cancel{\sum_{i=1}^{4} c_i \left[e^{\lambda_i t} b \right.}$$

$$= \underbrace{\begin{bmatrix} e^{\lambda_+ t}\vec{b}_1 & e^{\bar{\lambda}_+ t}\vec{b}_1 & e^{\lambda_- t}\vec{b}_2 & e^{\bar{\lambda}_- t}\vec{b}_2 \\ \lambda_+ e^{\lambda_+ t}\vec{b}_1 & \bar{\lambda}_+ e^{\bar{\lambda}_+ t}\vec{b}_1 & \lambda_- e^{\lambda_- t}\vec{b}_2 & \bar{\lambda}_- e^{\bar{\lambda}_- t}\vec{b}_2 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

"Wronskian matrix" for 4 independent solutions of 2nd order 2x2 DE system

$$\vec{X} = c_1 e^{\lambda_+ t}\vec{b}_1 + c_2 e^{\bar{\lambda}_+ t}\vec{b}_1 + c_3 e^{\lambda_- t}\vec{b}_2 + c_4 e^{\bar{\lambda}_- t}\vec{b}_2$$

$$= \underbrace{\underbrace{(c_1 e^{\lambda_+ t} + c_2 e^{\bar{\lambda}_+ t})}_{\tilde{c}_1}}_{e^{-\frac{t}{2}}(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t)} \vec{b}_1 + \underbrace{\underbrace{(c_3 e^{\lambda_- t} + c_4 e^{\bar{\lambda}_- t})}_{\tilde{c}_3}}_{e^{-t/2}(c_3 \cos\sqrt{11}t + c_4 \sin\sqrt{11}t)} \vec{b}_2$$

Re and Im parts of 1st and 3rd columns gives 4 independent real vector solution functions

For example

$$e^{\lambda_+ t}\vec{\mathcal{b}}_1 = e^{-t/2}(\cos\tfrac{\sqrt{3}}{2}t + i\sin\tfrac{\sqrt{3}}{2}t)\begin{bmatrix} 1 \\ 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} = e^{-t/2}\begin{bmatrix} \cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t \\ \cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t \\ -\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}\cos\frac{\sqrt{3}}{2}t + i\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right) \\ -\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}\cos\frac{\sqrt{3}}{2}t + i\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \\ e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t \\ e^{-\frac{t}{2}}\left(-\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}\cos\frac{\sqrt{3}}{2}t\right) \\ e^{-\frac{t}{2}}\left(-\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}\cos\frac{\sqrt{3}}{2}t\right) \end{bmatrix} + i\begin{bmatrix} e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t \\ e^{-t/2}\sin\frac{\sqrt{3}}{2}t \\ e^{-t/2}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right) \\ e^{-t/2}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}\right) \end{bmatrix} = \vec{\chi}_1 + i\vec{\chi}_2$$

get $\vec{\chi}_3, \vec{\chi}_4$ by $3 \rightarrow 11$.

best to let Maple handle all these calculations method of undetermined coefficients for constant coefficient linear systems : SOLUTION (3)

$$x_1'' + x_1' + 2x_1 - x_2 = 0$$
$$x_2'' + x_2' + -x_1 + 2x_2 = \cos 2t$$

reduction of order leads to a complex matrix of eigenvectors and a complex driving function in the uncoupled complex variables. to avoid complex arithmetic one can work in the original variables

$$\left.\begin{aligned} x_{1p} &= d_1\cos 2t + d_2 \sin 2t \\ x_{2p} &= d_3 \cos 2t + d_4 \sin 2t \end{aligned}\right\}$$ trial solution of appropriate type for all variables

just backsubstitute and solve for coefficients to get response solution

$$-4(d_1\cos 2t + d_2 \sin 2t)$$
$$+2(-d_1 \sin 2t + d_2 \cos 2t)$$
$$+2(d_1 \cos 2t + d_2 \sin 2t)$$
$$-(d_3 \cos 2t + d_4 \sin 2t)$$

$$\underbrace{(-4d_1 + 2d_2 + 2d_1 - d_3)}_{\underbrace{-2d_1 + 2d_2 - d_3}_{=0}} \cos 2t + \underbrace{(-4d_2 - 2d_1 + 2d_2 - d_4)}_{\underbrace{-2d_1 - 2d_2 - d_4}_{=0}} \sin 2t = 0$$

$$-4(d_3 \cos 2t + d_4 \sin 2t)$$
$$+2(-d_3 \sin 2t + d_4 \cos 2t)$$
$$-(d_1 \cos 2t + d_2 \sin 2t)$$
$$+2(d_3 \cos 2t + d_4 \sin 2t)$$

$$\underbrace{(-4d_3 + 2d_4 - d_1 + 2d_3)}_{\underbrace{-2d_3 + 2d_4 - d_1}_{=1}} \cos 2t + \underbrace{(-4d_4 - 2d_3 - d_2 + 2d_4)}_{\underbrace{-2d_4 - 2d_3 - d_2}_{=0}} \sin 2t = \cos 2t$$

matrix form of system:

$$\begin{bmatrix} -2 & 2 & -1 & 0 \\ -2 & -2 & 0 & -1 \\ -1 & 0 & -2 & 2 \\ 0 & -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{solve}} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \frac{1}{65} \begin{bmatrix} -1 \\ -8 \\ -14 \\ 18 \end{bmatrix}$$

row reduce or use matrix inverse

$$\downarrow$$

$$\begin{bmatrix} x_{1p} \\ x_{2p} \end{bmatrix} = \frac{1}{65} \begin{bmatrix} -\cos 2t - 8 \sin 2t \\ -14 \cos 2t + 18 \sin 2t \end{bmatrix}$$