

Exercise: coupled damped harmonic oscillators etc worked

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} + \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos 2t \\ \cos 2t \end{bmatrix} \rightarrow \text{decoupled scalar eqns}$$

$$y_1'' + y_1' + y_1 = \frac{1}{2} \cos 2t$$

$$y_1 \sim e^{rt} \rightarrow r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm i\sqrt{3}}{2}$$

$$e^{rt} = e^{-\frac{1}{2}t} e^{\pm i\frac{\sqrt{3}}{2}t} = e^{-\frac{1}{2}t} (\cos \frac{\sqrt{3}}{2}t \pm i \sin \frac{\sqrt{3}}{2}t)$$

$$\hookrightarrow e^{-t/2} \cos \frac{\sqrt{3}}{2}t, e^{-t/2} \sin \frac{\sqrt{3}}{2}t$$

$$y_{1p} = e^{-\frac{1}{2}t} (c_5 \cos \frac{\sqrt{3}}{2}t + c_6 \sin \frac{\sqrt{3}}{2}t)$$

$$1 [y_{1p} = c_5 \cos 2t + c_6 \sin 2t]$$

$$1 [y_{1p}' = -2c_5 \sin 2t + 2c_6 \cos 2t]$$

$$1 [y_{1p}'' = -4c_5 \cos 2t - 4c_6 \sin 2t]$$

$$y_{1p}'' + y_{1p}' + y_{1p} = [(t-4) c_5 + 2c_6] \cos 2t + [-2c_5 + (4-t)c_6] \sin 2t$$

$$= (-3c_5 + 2c_6) \cos 2t + (-2c_5 - 3c_6) \sin 2t$$

$$= \frac{1}{2} \cos 2t$$

$$\begin{aligned} -3c_5 + 2c_6 &= \frac{1}{2} \\ -2c_5 - 3c_6 &= 0 \end{aligned}$$

$$\begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_5 \\ c_6 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} y_2 \\ 0 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$y_{1p} = \frac{1}{20} \begin{bmatrix} -3 \cos 2t + 2 \sin 2t \end{bmatrix}$$

$$y_2'' + y_2' + 3y_2 = \frac{1}{2} \cos 2t$$

$$y_2 \sim e^{rt} \rightarrow r^2 + r + 3 = 0$$

$$r = -\frac{1 \pm i\sqrt{11}}{2}$$

$$e^{rt} = e^{-\frac{1}{2}t} e^{\pm i\frac{\sqrt{11}}{2}t} = e^{-\frac{1}{2}t} (\cos \frac{\sqrt{11}}{2}t \pm i \sin \frac{\sqrt{11}}{2}t)$$

$$\hookrightarrow e^{-t/2} \cos \frac{\sqrt{11}}{2}t, e^{-t/2} \sin \frac{\sqrt{11}}{2}t$$

$$y_{2h} = e^{-\frac{1}{2}t} (c_7 \cos \frac{\sqrt{11}}{2}t + c_8 \sin \frac{\sqrt{11}}{2}t)$$

$$3 [y_{2p} = c_7 \cos 2t + c_8 \sin 2t]$$

$$1 [y_{2p}' = -2c_7 \sin 2t + 2c_8 \cos 2t]$$

$$1 [y_{2p}'' = -4c_7 \cos 2t - 4c_8 \sin 2t]$$

$$y_{2p}'' + y_{2p}' + 3y_{2p} = [3-4] c_7 + 2c_8 \cos 2t$$

$$+ [-2c_7 + (3-4)c_8] \sin 2t$$

$$= (-c_7 + 2c_8) \cos 2t + (-2c_7 - c_8) \sin 2t$$

$$= \frac{1}{2} \cos 2t$$

$$\begin{aligned} -c_7 + 2c_8 &= \frac{1}{2} \\ -2c_7 - c_8 &= 0 \end{aligned}$$

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} c_7 \\ c_8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_7 \\ c_8 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$y_{2p} = \frac{1}{10} (-\cos 2t + 2 \sin 2t)$$

$$\vec{x} = \underbrace{(y_{1h} + y_{1p})}_{\vec{B}_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{(y_{2h} + y_{2p})}_{\vec{B}_2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{y_{1h} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_{2h} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{X}_h} + \underbrace{\frac{1}{20} (-3 \cos 2t + 2 \sin 2t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{10} (-\cos 2t + 2 \sin 2t) \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{X}_p}$$

$$\frac{-3c_5 + 13}{2c_5} = \frac{-18}{26.5} = \frac{-1}{13}$$

$$\frac{2.5 - 2c_5}{2c_5} = \frac{-16}{26.5} = -\frac{8}{13}$$

$$\frac{-3c_5 - 13}{2c_5} = \frac{-28}{26.5} = -\frac{14}{13}$$

$$\frac{2.5 + 2c_5}{2c_5} = \frac{36}{26.5} = \frac{18}{13}$$

$$\vec{x}_p = \begin{bmatrix} \left(\frac{-3}{26} + \frac{1}{10} \right) \cos 2t + \left(\frac{2}{26} - \frac{2}{10} \right) \sin 2t \\ \left(\frac{3}{26} - \frac{1}{10} \right) \cos 2t + \left(\frac{2}{26} + \frac{2}{10} \right) \sin 2t \end{bmatrix}$$

$$= \frac{1}{65} \begin{bmatrix} -\cos 2t - 8 \sin 2t \\ -14 \cos 2t + 18 \sin 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t/2} (c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t) - e^{-t/2} (c_3 \cos \frac{\sqrt{11}}{2}t + c_4 \sin \frac{\sqrt{11}}{2}t) & + \frac{1}{65} (-\cos 2t - 8 \sin 2t) \\ e^{-t/2} (c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t) + e^{-t/2} (c_3 \cos \frac{\sqrt{11}}{2}t + c_4 \sin \frac{\sqrt{11}}{2}t) & + \frac{1}{65} (-14 \cos 2t + 18 \sin 2t) \end{bmatrix}$$

↑

Maple has opposite sign here since it uses $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
making first nonzero entry of eigenvector 1
instead of last as in row reduction choice.