Exercise: coupled damped harmonic oscillators with "compatible" damping

coupled: $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} \leftrightarrow \vec{x}'' + \vec{x}' = A\vec{x} + \vec{F}$

(unit matrix coefficient; $-A$)

$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, $B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2$, $\vec{y} = B^{-1}\vec{x}$

$A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$, $B^{-1}\vec{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \cos 2t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cos 2t \\ \frac{1}{2}\cos 2t \end{bmatrix}$

decoupled: $\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} + \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cos 2t \\ \frac{1}{2}\cos 2t \end{bmatrix}$

a) Write down the two scalar DEs for y_1 and y_2.

b) Find their homogeneous solutions. (transient)

c) Find their particular solutions. (steady state response)

d) Go back: $\vec{x} = \underbrace{(y_{1h} + y_{1p})}_{y_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{(y_{2h} + y_{2p})}_{y_2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{\left(y_{1h}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_{2h}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)}_{\vec{x}_h} + \underbrace{\left(y_{1p}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_{2p}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)}_{\vec{x}_p}$

simplify this linear combination

e) Write down gen soln: $\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ combining all the terms.

f) Write down the scalar DEs for x_1, x_2 and solve with Maple. compare the particular solutions.