Reduction of order worked example

Given $\boxed{x'' + 3x' + 2x = \cos t \quad , \quad x(0) = 1,\ x'(0) = 2}$

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$ initial data vector

so that $x_1' = x_2$

$x_2' + 3x_2 + 2x_1 = \cos t \quad , \quad x_1(0) = 1,\ x_2(0) = 2$

Thus $\quad x_1' = x_2$

$\qquad x_2' = -2x_1 - 3x_2 + \cos t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \cos t \end{bmatrix}}_{\underline{f}}$$

Matrix form: $\underline{x}' = A\underline{x} + \underline{f} \quad , \quad \underline{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ nonhomogeneous linear first order constant coefficient system

$$\boxed{A: \ \lambda = -1, -2 \qquad B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix},\ B^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix},\ A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}$$

Change variables: $\underline{x} = B\underline{y}\ ,\ \underline{y} = B^{-1}\underline{x}$

$$\underline{y}' = B^{-1}\underline{x}' = B^{-1}(A\underline{x} + \underline{f}) = B^{-1}(AB\underline{y} + \underline{f}) = A_B\underline{y} + \underbrace{B^{-1}\underline{f}}_{\text{new components of driving vector function}}$$

TODO:

1) Evaluate $B^{-1}\underline{f}$ and solve decoupled equations $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + B^{-1}\underline{f}$ using first order linear technique with integrating factors

   $\left(e^{pt}\left[y' + py = q\right] \rightarrow (ye^{pt})' = qe^{pt} \ \text{etc.} \right)$ (e^{pt})

   or use method of undetermined coefficients

2) Reconstruct $\underline{x} = B\underline{y}$.

3) Solve initial conditions for $\underline{x}$.

4) Evaluate x_1 and x_2.

5) Compare with solution $x = \frac{7}{2}e^{-t} - \frac{13}{5}e^{-2t} + \frac{1}{10}(\cos t + 3\sin t)$ by evaluating x' and comparing your solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$ with this pair of functions.

   They should agree!