

reduction of order

$x_1'' + 3x_1' + 2x_1 = \cos t$
 $x_1(0) = 1, x_1'(0) = 2$

initial data
or
"state" vector:

$$\begin{bmatrix} x_1 \\ x_1' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}$$

its second component
can be taken as a new variable: $x_2 = x_1'$ so that $x_2' = x_1''$

rewrite: $x_2' + 3x_2 + 2x_1 = \cos t$

one second order equation
becomes 2 first order equations

initial conditions: $\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

solve
for x_2'

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 3x_2 + \cos t \end{aligned}$$

write
in
matrix
form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

$$\vec{x}' = A\vec{x} + \vec{F}$$

initial data
or "state"
vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \vec{x} \\ \vec{x}' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x}$$

so we have 2 new variables

$$x_3 = x_1' \text{ and } x_4 = x_2'$$

whose derivatives are $x_3' = x_1''$ and $x_4' = x_2''$

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= -2x_1 + x_2 - x_3 \\ x_4' &= x_1 - 2x_2 - x_4 + (\cos 2t) \end{aligned}$$

rewrite
in matrix
form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 2t \end{bmatrix}$$

$$\vec{x}' = A\vec{x} + \vec{F}$$

In general, for 2nd order vector variable:

$$\vec{x}'' + A_1 \vec{x}' + A_0 \vec{x} = \vec{F}$$

$$\begin{bmatrix} \vec{x} \\ \vec{x}' \end{bmatrix}' = \begin{bmatrix} I_2 & A_1 \\ -A_0 & -A_1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{x}' \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{F} \end{bmatrix}$$

$\underbrace{\text{block matrix}}_{\text{containing both coefficient matrices.}}$

$$\begin{bmatrix} \vec{x} \\ \vec{x}' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

first order
variables

Reduction of order worked example

Given

$$x'' + 3x' + 2x = \cos t, \quad x(0) = 1, x'(0) = 2$$

$$x_2' + 3x_2 + 2x_1 = \cos t, \quad x_1(0) = 1, x_2(0) = 2$$

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$ initial data vector
so that $x_1' = x_2$

Thus

$$x_1' = x_2$$

$$x_2' = -2x_1 - 3x_2 + \cos t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \cos t \end{bmatrix}}_f$$

Matrix form:

$$\underline{x}' = A\underline{x} + \underline{f}, \quad \underline{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

nonhomogeneous linear first order constant coefficient system

Change variables:

$$\underline{x} = B\underline{y}, \quad \underline{y} = B^{-1}\underline{x}$$

$$A: \lambda = -1, -2$$

$$B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\underline{y}' = B^{-1}\underline{x}' = B^{-1}(A\underline{x} + \underline{f}) = B^{-1}(A B \underline{y} + \underline{f}) = A_B \underline{y} + B^{-1}\underline{f}$$

new components of driving vector function

TODO:

1) Evaluate $B^{-1}\underline{f}$

and solve decoupled equations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + B^{-1}\underline{f}$$

using first order linear technique with integrating factors

2) Reconstruct $\underline{x} = B\underline{y}$.

$$\left(e^{pt} [y' + py = q] \rightarrow (ye^{pt})' = qe^{pt} \right)$$

etc.

3) Solve initial conditions for \underline{x} .

or use method of undetermined coefficients

4) Evaluate x_1 and x_2 .

5) Compare with solution $x = \frac{7}{2}e^{-t} - \frac{13}{5}e^{-2t} + \frac{1}{10}(\cos t + 3 \sin t)$

by evaluating x' and comparing your solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$ with this pair of functions.

They should agree!