power series and differential equations : an aside

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

A differential equation, once solved for its highest derivative, is essentially a "recurrence relation" that can be used by successive differentiation and backsubstitution to determine that highest derivative and all higher derivatives in terms of the values of the unknown function and its lower derivatives (i.e., the initial data variables) at any value of the independent variable.

This means that given initial data at $x = x_0$, one can evaluate $y(x_0)$ and all of its derivatives there and hence obtain a power series representation of the unknown function $y(x)$ in which the initial data numbers are freely specifiable, i.e., an n-parameter family of solution functions, valid within some radius of convergence. This is why every nth order DE must have n arbitrary constants in its general solution.

example IVP: $y'' = -y, \quad y(0) = y_0, \quad y'(0) = v_0$

$$y'' = -y \xrightarrow{d/dx} y^{(3)} = -y' \xrightarrow{d/dx}$$
$$y^{(4)} = -y'' = (-1)^2 y \xrightarrow{d/dx} y^{(5)} = (-1)^2 y' \xrightarrow{d/dx}$$
$$y^{(6)} = (-1)^2 y'' = (-1)^3 y \xrightarrow{d/dx} y^{(7)} = (-1)^3 y' \xrightarrow{d/dx} \ldots$$
$$y^{(2n)} = (-1)^n y \xrightarrow{d/dx} y^{(2n+1)} = (-1)^n y'$$

Each time the derivative reaches y'', we use the D.E. to reexpress it in terms of y, so no derivative higher than y' need ever appear.

Taylor expansion:

$$y(x) = \sum_{m=0}^{\infty} y^{(m)}(0)\frac{x^m}{m!} = \sum_{n=0}^{\infty} y^{(2n)}(0)\frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} y^{(2n+1)}(0)\frac{x^{2n+1}}{(2n+1)!} \qquad \text{even \& odd terms}$$

$$= \sum_{n=0}^{\infty} (-1)^n y(0)\frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n y'(0)\frac{x^{2n+1}}{(2n+1)!} = y_0\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right) + v_0\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)$$

$$= y_0 \cos x + v_0 \sin x \qquad \text{for } -\infty < x < \infty. \quad \text{(unique solution of IVP)}$$

Every DE textbook has a chapter on power series methods for solving DEs.
The above calculation could be done alternatively by assuming $y(x)$ has a power series solution and using the DE to obtain a recurrence relation among its coefficients.

see example 4, p. 628 Edwards and Penney DE & LinAlg Second Edition which redoes the above example using this technique.

Boundary conditions instead of initial conditions (power series motivation)

Suppose we consider imposing 2 conditions which are not initial conditions to determine the 2 arbitrary constants in the general solution of a second order DE. We could fix the value of the unknown at the endpoints of an interval for example. These are called boundary conditions since they are imposed on the boundary of the interval where we want to solve the DE.

vibration profiles for a unit length guitar string

$$y'' + \omega^2 y = 0\,, \qquad y(0) = 0 = y(1)$$

max displacement of string from straight position is $y(x)$, but fixed on endpoints

general solution

$y = c_1 \cos \omega x + c_2 \sin \omega x$

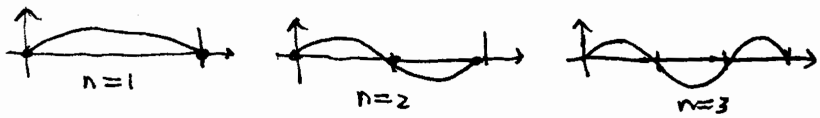
$y(0) = c_1 = 0$

$y(1) = c_1 \cos\omega + c_2 \sin\omega = 0$

$c_2 \neq 0 \to \sin\omega = 0 \to \omega = n\pi,\ n = 1, 2, \dots$

(if $c_1 = 0$ and $c_2 = 0$, $y = 0$ pretty boring, eh?)

This requires only the sine term & forces the frequency ω to be an integer multiple of π, leading to the socalled "harmonics"



etc. . .

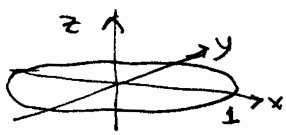
and also require the vibration frequency of the oscillating profile in time be integer multiples of π.

An arbitrary vibration profile can be constructed from a linear combination of all of these individual harmonics

$$\underbrace{y(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x}_{\text{"Fourier analysis"}} \xrightarrow[\text{time when released}]{\text{oscillate in}} y(x,t) = \sum_{n=1}^{\infty} A_n \sin n\pi x \cos n\pi t$$

$y(x,t)$: solution of partial differential equation since now 2 independent variables "wave equation" (standing waves here)

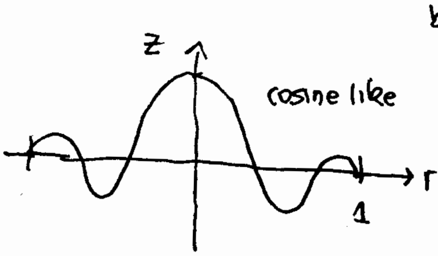
If one considers the 2-dimensional generalization to vibrations of a drum head,



↕ vibrates in z direction $z(x,y)$

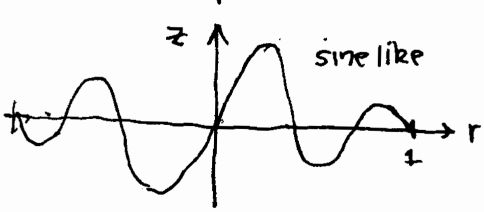
$z = 0$ on circular boundary

the radial behavior of the corresponding profile functions $z(x,y)$ require larger values in the center to compensate for the smaller contribution to the total area of the drum head (equal tension on a smaller area leads to bigger displacement, sort of...)



so one finds decaying sinusoidal like profile functions.

This is where the famous Bessel functions arise analogous to the sine (and cosine) functions in the 1-dimensional problem.



$\boxed{\text{power series methods}}$ are necessary to find a representation of these new functions.