optional aside on phase spaces and reduction of order (read at your own risk!)

particle motion definitions

$\vec{v} = \vec{r}\,'$

$\vec{a} = \vec{v}\,' = \vec{r}\,''$

Newton's law

$\vec{F} = m\vec{a}$

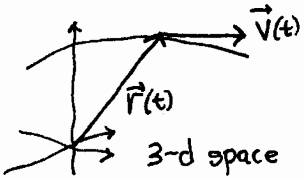
usual approach $\longrightarrow$ $m\vec{r}\,'' = \vec{F}$ (2nd order DE)

reduction of order

substitute $\vec{r}\,'' \to \vec{v}\,'$, $\vec{r}\,' \to \vec{v}$ in DE, add equation $\vec{r}\,' = \vec{v}$ to double number eqns, but halve order from 2nd order to 1st order:

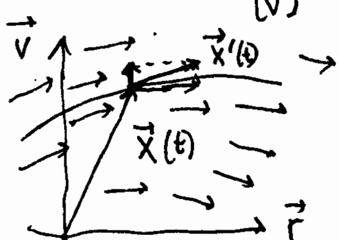
$$\begin{aligned}\vec{r}\,' &= \vec{v}\\ m\vec{v}\,' &= \vec{F}\end{aligned} \quad \text{or} \quad \begin{aligned}\vec{r}\,' &= \vec{v}\\ \vec{v}\,' &= \vec{F}/m\end{aligned} \quad \text{or} \quad \underline{x}\,' = \begin{bmatrix}\vec{r}\\ \vec{v}\end{bmatrix}' = \underbrace{\begin{bmatrix}\vec{v}\\ \vec{F}/m\end{bmatrix}}$$

vector field on velocity phase space:



state of particle at time t $\Rightarrow$ state vector $\vec{x} = \begin{bmatrix}\vec{r}\\ \vec{v}\end{bmatrix}$

soln $\vec{x}(t)$ is a curve which connects up the arrows, solves 1st order vector DE.



once you solve these equations for $\vec{x}(t)$, ignoring the second half of the vector gives your solution $\vec{r}(t)$ of Newton's law.

state or phase space = 6d (velocity phase space)

Dangerous Bend (only for those with some curiosity about Quantum Mechanics)

In physics we also use "momentum phase space". For 1d motion with Hooke's law

state vector $\begin{bmatrix}x\\ p\end{bmatrix} = \begin{bmatrix}x\\ mv\end{bmatrix}$

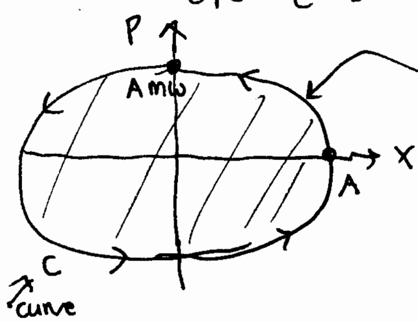
$mx'' = -kx \to x'' + \omega^2 x = 0$ where $\omega = \sqrt{\frac{k}{m}}$

solns:

$x = A\cos(\omega t - \delta)$

$v = -A\omega\sin(\omega t - \delta)$

$p = mv = -Am\omega\sin(\omega t - \delta)$



$$\boxed{\begin{gathered}\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\\ \text{area} = \pi ab\\ \text{calc 2 result}\end{gathered}}$$

note: $\frac{x^2}{A^2} + \frac{p^2}{A^2m^2\omega^2} = \cos^2(\omega t - \delta) + \sin^2(\omega t - \delta) = 1$

ellipse with semiaxes A, $Am\omega$

path in momentum phase space is an ellipse for simple harmonic motion

In the naive beginnings of Quantum Mechanics, Bohr said the system could only exist in quantum states at energies for which the area of this ellipse was an integer multiple of the fundamental unit h of "action" (Planck's constant)

Action integral $=$ "$\oint_C p\,dx$" $=$ Area of ellipse $= \pi(A)(Am\omega) = \pi A^2 m\omega = nh \quad n = 0,1,2,\ldots$

Then the energy is quantized:

$$E = \underbrace{\frac{p^2}{2m}}_{\text{kinetic} = \frac{1}{2}mv^2} + \underbrace{\frac{1}{2}kx^2}_{\text{potential energy}} = \frac{(-Am\omega\sin(\omega t - \delta))^2}{2m} + \frac{\overset{= m\omega^2}{k}}{2}[A\cos(\omega t - \delta)]^2 = \frac{A^2m\omega^2}{2}[\sin^2(\omega t - \delta) + \cos^2(\omega t - \delta)]$$

$$= \frac{A^2m\omega^2}{2} = (A^2m\omega)\frac{\omega}{2} = n\frac{h}{2\pi}\omega \qquad \frac{h}{2\pi} \equiv \hbar$$

$$\boxed{= n\hbar\omega}$$

Later it turned out to be $E = (n + \frac{1}{2})\hbar\omega$, $n = 0,1,2$ so lowest energy is "zero pt energy" $\frac{1}{2}\hbar\omega$ due to uncertainty principle.