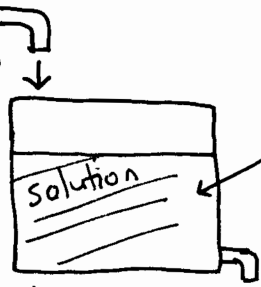
mixture problems = solution concentration problems

(example of how one comes up with a DE from rate information)

input rate concentration r_i, c_i constants (in)



assume thoroughly mixed soln

(out) output rate concentration $r_o, c_o(t)$

- amount of solute $X(t)$
- volume of solution $V(t)$
- concentration of solute = outgoing concentration $\quad c_o(t) = \dfrac{X(t)}{V(t)}$

UNITS:

mass/weight: g or lb

liquid volume: L or gal

$\dfrac{\text{mass-weight}}{\text{liquid volume}}$: $\dfrac{g}{L}, \dfrac{lb}{gal}$

*

liquid flow rate r: $\dfrac{\text{liquid volume}}{\text{time}}$ $\dfrac{L}{s}, \dfrac{gal}{h}$ etc

product rc: $\dfrac{\text{mass-weight}}{\text{time}}$ $\dfrac{g}{s}, \dfrac{lb}{h}$ etc

during time interval Δt:
$$\Delta X = (\text{amount in}) - (\text{amount out})$$
$$\approx \underbrace{(r_i c_i)}_{\text{in rate} = \frac{\text{amount}}{\text{time}}} \Delta t - \underbrace{(r_o c_o)}_{\text{out rate} = \frac{\text{amount}}{\text{time}}} \Delta t$$

$$\frac{\Delta X}{\Delta t} \approx r_i c_i - r_o c_o(t)$$

$\lim_{\Delta t \to 0}$

$$\frac{dX(t)}{dt} = r_i c_i - r_o \underbrace{c_o(t)}_{= \frac{X(t)}{V(t)} = \frac{X(t)}{V_0 + (r_i - r_o)t}}$$

Volume

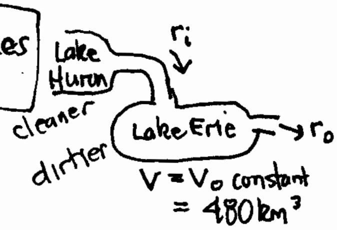
$V_0 = V(0)$ initial volume

$\dfrac{dV}{dt} = \underbrace{(r_i - r_o)}_{\text{net flow rate}}$

$V = V_0 + (r_i - r_o)t$ linearly increases

linear DE
$$\boxed{\frac{dX}{dt} + \frac{r_o}{V_0 + (r_i - r_o)t} X = r_i c_i \,, \quad X(0) = X_0}$$

Great Lakes Pollution



$r = r_i = r_o = 350\, km^3/yr$ (no accumulation of water)

Initial condition $(t=0)$: initial pollution concentration of Lake Erie is five times that of Lake Huron: $c_o(0) = 5c_i$ (not given)

cleaner water mixes, dilutes pollution.

how long till reduce concentration level to $c_o(t) = 2c_i$?

strategy: First calculate $X(t)$ in terms of c_i, then calculate $c_o(t) = X(t)/V_0$ and set equal to $2c_i$, solve for t.

initial concentration $\dfrac{X_0}{V_0} = \dfrac{X(0)}{V_0} = 5c_i$ so I.C.: $X(0) = 5c_iV_0$ (initial amount of pollutants)

$$e^{\frac{r}{V_0}t}\left[\frac{dx}{dt} + \frac{r}{V_0}x = rc_i\right] \to \frac{d}{dt}\left(xe^{\frac{r}{V_0}t}\right) = rc_i e^{\frac{r}{V_0}t} \to xe^{\frac{r}{V_0}t} = \frac{rc_i}{r/V_0} e^{\frac{r}{V_0}t} + C_1 \to x = c_iV_0 + C_1 e^{-\frac{r}{V_0}t}$$

$e^{\frac{r}{V_0}t}$ $\qquad \dfrac{rc_i}{r/V_0} \to c_iV_0$

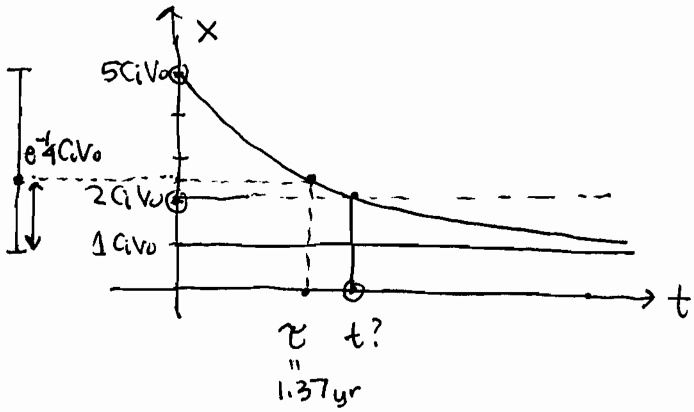
$5c_iV_0 = c_iV_0 + C_1$

$\to C_1 = 4c_iV_0$

soln IVP:
$$\boxed{\begin{aligned} x &= c_iV_0 + 4c_iV_0 e^{-\frac{r}{V_0}t} \\ &= c_iV_0(1 + 4e^{-\frac{r}{V_0}t}) \end{aligned}}$$

Set $x = 2c_iV_0$:
$$c_iV_0(1 + 4e^{-\frac{r}{V_0}t}) = 2c_iV_0 \to 1 + 4e^{-\frac{r}{V_0}t} = 2 \to e^{-\frac{r}{V_0}t} = \frac{1}{4}$$
$$e^{\frac{r}{V_0}t} = 4 \quad \frac{r}{V_0}t = \ln 4 \quad t = \frac{V_0}{r}\ln 4 = \frac{(480\,km^3)}{(350\,km^3/yr)}\ln 4 \approx 1.9\,yr$$

E&P3 1.5.37 Great Lakes Pollution : the diagram & τ



exponential decay of difference $\sim e^{-\frac{r}{V_0}t}$

$$\tau = \frac{V_0}{r} = \frac{480\ \text{km}^3}{350\ \text{km}^3/\text{yr}} = 1.37\,\text{yr}$$

(time to fill empty lake at this flow rate)

We found $t = 1.9$ yr, a bit longer than τ as the diagram suggests.

If we waited till the difference was 1% of the initial difference $4C_iV_0$, it would be $t = 4.6\tau = 6.3$ yr.

Knowing the difference decays exponentially, we could have answered the final question immediately: how long does the level reduce from 5 to 2 times the asymptotic value of $1\ C_iV_0$?

$$\underbrace{4}_{\substack{\text{initial}\\\text{difference}\\\text{factor}\\=5-1}}e^{-t/\tau} = \underbrace{1}_{\substack{\text{final}\\\text{difference}\\\text{factor}\\=2-1}} \longrightarrow -\frac{t}{\tau} = \ln\frac{1}{4} \to t = \tau\ln 4 = (1.37)(1.386)$$

$$= 1.92 \approx 1.9\,\text{yr}$$

$\left(t = \tau(-\ln\frac{1}{4}) \right)$

$\left(e^{-t/\tau} = \frac{1}{4} \right.$ take ln both sides$\left.\right)$