Matrix inverse as simultaneous soln of simultaneous linear systems

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}}_{A^{-1}?} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftrightarrow \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

same coefficient matrices, different RHS vectors

$$\left[\begin{array}{cc|c|c} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

augment with both RHS vectors

if A reduces to I then:

$$\Big\downarrow \text{rref}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & u_1 & v_1 \\ 0 & 1 & u_2 & v_2 \end{array}\right] = \left[\begin{array}{cccc} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \end{array}\right]$$

$= A^{-1}$!

ignore this column

solve: $x_1 = u_1$, $x_2 = u_2$

or ignore first column, solve: $y_1 = v_1$, $y_2 = v_2$

in other words if left half reduces to the identity matrix, the right half is the inverse!

This clearly extends to any $n \times n$ matrix.