Matrix Inverse by Row Reduction

Goal: for a square matrix A find inverse A^{-1} such that $A^{-1}A = I = AA^{-1}$.

Fact: each row op in the row reduction process can be accomplished by first doing it to the left identity matrix and then left multiplying the target matrix by the result.

$$R_1\leftrightarrow R_2 \qquad \overset{I_2}{\begin{bmatrix}1&0\\0&1\end{bmatrix}} \to \overbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}^{E_1}\overbrace{\begin{bmatrix}3&2\\1&2\end{bmatrix}}^{A} = \begin{bmatrix}1&2\\3&2\end{bmatrix} \qquad \begin{bmatrix}0&1\\1&0\end{bmatrix}\overbrace{\begin{bmatrix}3&2&1&0\\1&2&0&1\end{bmatrix}}^{\langle A|I_2\rangle} = \begin{bmatrix}1&2&0&1\\3&2&1&0\end{bmatrix}$$

$$R_2\to R_2-3R_1 \qquad \begin{bmatrix}1&0\\0&1\end{bmatrix} \to \overbrace{\begin{bmatrix}1&0\\-3&1\end{bmatrix}}^{E_2}\overbrace{\begin{bmatrix}1&2\\3&2\end{bmatrix}}^{E_1A} = \begin{bmatrix}1&2\\0&-4\end{bmatrix} \qquad \begin{bmatrix}1&0\\-3&1\end{bmatrix}\begin{bmatrix}1&2&0&1\\3&2&1&0\end{bmatrix} = \begin{bmatrix}1&2&0&1\\0&-4&1&-3\end{bmatrix}$$

$$R_2\to -\tfrac{1}{4}R_2 \qquad \begin{bmatrix}1&0\\0&1\end{bmatrix} \to \overbrace{\begin{bmatrix}1&0\\0&-1/4\end{bmatrix}}^{E_3}\overbrace{\begin{bmatrix}1&2\\0&-4\end{bmatrix}}^{E_2E_1A} = \begin{bmatrix}1&2\\0&1\end{bmatrix} \qquad \begin{bmatrix}1&0\\0&-1/4\end{bmatrix}\begin{bmatrix}1&2&0&1\\0&-4&1&-3\end{bmatrix} = \begin{bmatrix}1&2&0&1\\0&1&-1/4&3/4\end{bmatrix}$$

$$R_1\to R_1-2R_2 \qquad \begin{bmatrix}1&0\\0&1\end{bmatrix} \to \overbrace{\begin{bmatrix}1&-2\\0&1\end{bmatrix}}^{E_4}\overbrace{\begin{bmatrix}1&2\\0&1\end{bmatrix}}^{E_3E_2E_1A} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \qquad \begin{bmatrix}1&-2\\0&1\end{bmatrix}\begin{bmatrix}1&2&0&1\\0&1&-1/4&3/4\end{bmatrix} = \Big[\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I_2}\ \underbrace{\begin{matrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{4}&\frac{3}{4}\end{matrix}}_{A^{-1}}\Big]$$

"elementary row matrices"

$\therefore \text{rref}(A) = I_2$

conclusion:

$$\underbrace{(E_4E_3E_2E_1)}_{\therefore A^{-1}}A = I_2 \to (E_4E_3E_2E_1)I_2 = \underbrace{E_4E_3E_2E_1}_{A^{-1}}$$

$$\therefore A^{-1} = \begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{4}&\frac{3}{4}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2&-2\\-1&3\end{bmatrix}$$

If we apply the same sequence of row ops to I_2, we get A^{-1}.

so $A \to \langle A|I_2\rangle \xrightarrow[\text{rref}]{} \langle I_2|A^{-1}\rangle$

same sequence of row ops will apply to any columns appended to A.

identify this half of augmented matrix as the inverse

Why useful?

$A^{-1}[A\vec{x} = \vec{b}]$

$\underbrace{\underbrace{A^{-1}A}_{I}\vec{x}}_{\vec{x}} = A^{-1}\vec{b}$

$\therefore\ \vec{x} = A^{-1}\vec{b}$ is the soln

CHECK:

$$A^{-1}A = \frac{1}{4}\begin{bmatrix}2&-2\\-1&3\end{bmatrix}\begin{bmatrix}3&2\\1&2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}6-2&4-4\\-3+3&-2+6\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4&0\\0&4\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix}3&2\\1&2\end{bmatrix}\frac{1}{4}\begin{bmatrix}2&-2\\-1&3\end{bmatrix} = \frac{1}{4}\begin{bmatrix}6-2&-6+6\\2-2&-2+6\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4&0\\0&4\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{aligned}3x_1+2x_2&=1\\x_1+2x_2&=2\end{aligned} \qquad \begin{bmatrix}3&2\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \to \begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2&-2\\-1&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2-4\\-1+6\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}-1/2\\5/4\end{bmatrix}$$

But if $\langle A|I\rangle \not\to \langle I|B\rangle$ then A^{-1} does not exist, ie, matrices which do not reduce to the identity have no inverse.

$\boxed{3.5}$ 0

For real numbers:

multiplicative identity: $1a = a1 = a$

multiplicative inverse: $a^{-1}a = aa^{-1} = 1$ (in fact $a^{-1} = 1/a$)

why useful:

$ax = b$ standard form for linear condition on x

$$\downarrow$$

$$\underbrace{a^{-1}(ax)}_{\text{associativity } \underbrace{(\underbrace{a^{-1}a}_{\text{identity } 1})\,x}_{x}} = a^{-1}b \longrightarrow x = a^{-1}b \quad (= b/a)$$

square matrices can be multiplied in either order.

seek: $A^{-1}A = AA^{-1} = I_n$ ← for $n\times n$ matrices A, A^{-1}

If matrix inverse exists:

$$A\vec{x} = \vec{b}$$

$$\downarrow$$

$$\left.\begin{aligned} &A^{-1}(A\vec{x}) = A^{-1}\vec{b} \\ &\quad \| \\ &\underbrace{(A^{-1}A)\vec{x}}_{\vec{x}} \\ &\quad \| \\ &\quad I \end{aligned}\right\} \quad \vec{x} = A^{-1}\vec{b}$$

obtain soln by matrix mult instead of row reduction

(see reverse side) → EXAMPLE HANDOUT $\left(\text{check: } \begin{aligned} 3(-\tfrac{1}{2}) + 2(\tfrac{5}{4}) &= -\tfrac{3}{2} + \tfrac{5}{2} = 1 \ \checkmark \\ 1(-\tfrac{1}{2}) + 2(\tfrac{5}{4}) &= -\tfrac{1}{2} + \tfrac{5}{2} = 2 \ \checkmark \end{aligned}\right)$

→ properties of inverse

1) $(A^{-1})^{-1} = ? = A$ ← $\underbrace{A^{-1}\underset{\sim}{A} = \underset{\sim}{A}A^{-1}}_{(A^{-1})^{-1}!} = I$

2) A^{-1}, B^{-1} exist, what is $(AB)^{-1}$? $\overset{\text{claim}}{=} B^{-1}A^{-1}$:

$$(B^{-1}A^{-1})(AB) = B^{-1}(\underbrace{A^{-1}A}_{I})B = B^{-1}B = I$$

works same on RHS.

HW extend to 3 factors.

0) UNIQUENESS: $BA = AB = I$, $CA = AC = I$?

$C = CI = C(AB) = (CA)B = IB = B$ ✓ must be same.