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# EPC4 7.5.14 dynamic damping


$k_1 = 50,\ k_2 = 10,\ m_1 = 1,\ \omega = 10$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -k_2/m_2 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 5\cos\omega t \\ 0 \end{bmatrix}}_{f}$$

$$A = \begin{bmatrix} -60 & 10 \\ 10/m_2 & -10/m_2 \end{bmatrix}$$

(force per unit mass)

Goal: Adjust $m_2$ to make the response displacement for $x_1$ zero. Use the method of undetermined coefficients for $x_1, x_2$ to evaluate the particular soln, which is the response vector.

Trial soln: $x = \langle x_1, x_2\rangle = \langle a\cos\omega t, b\cos\omega t\rangle$

$$x'' = -\omega^2 x = Ax + f$$

$$-\underbrace{(A + \omega^2 I)}\, x = f \longrightarrow x = -(A+\omega^2 I)^{-1} f$$

$\det(A - \underbrace{(-\omega^2)}_{\neq \text{ eigenvalue of } A} I) \neq 0$

insert

coefficients of $\cos\omega t$:

Explicitly:

$x_1'' + 60x_1 - 10x_2 = 5\cos 10t \longrightarrow -100a + 60a - 10b = 5$

$x_2'' - \frac{10}{m_2}x_1 + \frac{10}{m_2}x_2 = 0 \longrightarrow -100b - \frac{10}{m_2}a + \frac{10}{m_2}b = 0$

$$\underbrace{\begin{bmatrix} -40 & -10 \\ -\frac{10}{m_2} & \frac{10}{m_2}-100 \end{bmatrix}}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\begin{bmatrix} \frac{10}{m_2}-100 & 10 \\ \frac{10}{m_2} & -40 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix}}{500(8-1/m_2)} = \frac{\begin{bmatrix} 50(\frac{1}{m_2}-10) \\ 50/m_2 \end{bmatrix}}{500(8-1/m_2)}$$

det: $40(100 - \frac{10}{m_2}) - \frac{100}{m_2} = 500(8 - \frac{1}{m_2})$

if $m_2 = \frac{1}{8}$, $\det(A + \omega^2 I) = 0$ and $\lambda = -100$ is an eigenvalue corresponding to eigenfrequency $\omega = 10$.
In this case the trial solution must be multiplied by $t$ because the trial function is a homogeneous soln!

$$= \frac{\begin{bmatrix} \frac{1}{m_2}-10 \\ \frac{1}{m_2} \end{bmatrix}}{10(8-1/m_2)}$$

$\rightarrow a = 0 :\ m_2 = 1/10$

$b = \frac{100}{10(8-10)} = -\frac{1}{2}$

$x = \langle 0, -\frac{1}{2}\cos 10t\rangle$

mass 1 is fixed, mass 2 oscillates in opposite direction to the force on mass 1 to compensate for the force on it from the second spring.

Shortcut:
To make $x_1 = 0$ constant:
$0 + 60(0) - 10x_2 = 5\cos 10t$
so choose $x_2 = -\frac{1}{2}\cos 10t$ to exactly balance the applied force on the first mass

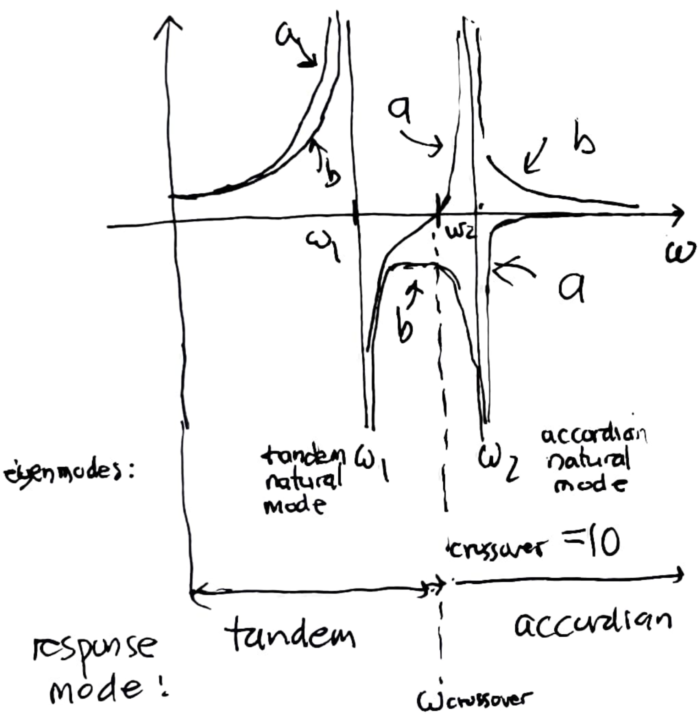
# EPC4 7.6.14 dynamic damping (2)

For $m = 1/8$:

$$A = \begin{bmatrix} -60 & 10 \\ 100 & -100 \end{bmatrix} \quad B \approx \begin{bmatrix} 0.57 & -0.17 \\ 1 & 1 \end{bmatrix}$$

| | |
|---|---|
| 6.53 | 10.84 |
| $\omega_1$ | $\omega_2$ |
| tandem mode | accordian mode |

If we study the response coefficients $[a(\omega), b(\omega)]$ for general frequency we find




This is typical for all of these 2 mass 2 spring systems.

The interval $0 \le \omega \le \omega_{crossover}$ containing the tandem natural mode frequency has a tandem mode response (same sign for $a(\omega)$, $b(\omega)$)

for $\omega > \omega_{crossover}$ containing the accordian mode frequency, the response mode is a accordian mode (opposite signs for $a(\omega)$, $b(\omega)$)

The first mass is stationary at the crossover frequency, where the second spring contracts or stretches to balance the force applied directly to the first mass.

$$\begin{cases} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a(\omega) \cos \omega t \\ b(\omega) \cos \omega t \end{pmatrix} \\ f_1 = 5 \cos \omega t \end{cases}$$

just a convenient factor
actual value unimportant.

It is the ratio of the applied force and the response amplitudes which matter.

$\boxed{\text{For fixed } M_1, k_1 \text{ one can always adjust the smaller mass-spring add on to zero out the oscillations in the first mass.}}$

EPC4 7.6.14 dynamic damping (more!)



$\boxed{\text{new } \omega_1, \omega_2}$:

$\omega_1^2 \equiv \frac{k_1}{m_1}$

$\omega_2^2 \equiv \frac{k_2}{m_2}$

separate natural frequencies (uncoupled)

$\gamma = \frac{m_2}{m_1}$ mass ratio

$$A = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{bmatrix} = \begin{bmatrix} -(\omega_1^2 + \gamma\omega_2^2) & \omega_2^2\gamma \\ \omega_2^2 & -\omega_2^2 \end{bmatrix}$$

$x = \begin{bmatrix} a \\ b \end{bmatrix} \cos\omega t$, $x'' = -\omega^2 x$

$$\downarrow$$

$$x'' - Ax = \begin{bmatrix} \omega_1^2 + \gamma\omega^2 - \omega^2 & -\gamma\omega_2^2 \\ -\omega_2^2 & \omega_2^2 - \omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \cos\omega t = \begin{bmatrix} f_1 \cos\omega t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \omega_1^2 + \gamma\omega_2^2 - \omega^2 & \gamma\omega_2^2 \\ -\omega_2^2 & \omega_2^2 - \omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\begin{bmatrix} \omega_2^2 - \omega^2 & -\gamma\omega_2^2 \\ \omega_2^2 & \omega_1^2 + \gamma\omega_2^2 - \omega^2 \end{bmatrix} \begin{bmatrix} f_1 \\ 0 \end{bmatrix}}{(\omega_1^2 + \gamma\omega_2^2 - \omega^2)(\omega_2^2 - \omega^2) + \gamma\omega_2^4} = \frac{f_1}{\mathcal{D}} \begin{bmatrix} \omega_2^2 - \omega^2 \\ \omega_2^2 \end{bmatrix}$$

$\mathcal{D} \nearrow$

$a = 0$ means $\omega^2 = \omega_2^2$, $\boxed{\omega = \omega_2}$

The second mass natural frequency is used to drive the first mass

or more precisely:

The second mass was chosen to make the second (uncoupled) mass spring natural frequency equal the driving frequency, which in turn is slighty less than the natural frequency of the combined system (accordian mode) where resonance occurs (when some damping is present).

Pushing this concrete problem with specific numbers to understand what lies behind it is the magic of mathematics.

But we can actually understand some obvious features of the frequency response functions $a(\omega)$, $b(\omega)$.

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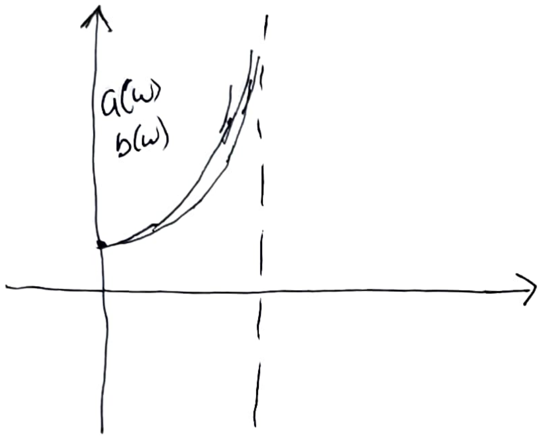
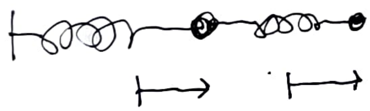



$\omega = 0$ zero frequency limit
$\cos(0) = 1$: constant force on first mass to right

both masses just move to new equilibrium equal distances to the right




since there is no third spring to compress the second spring

As $\omega$ increases from zero to very small values, the system simply slowly moves to new equilibrium values moment by moment so remains in phase with the slowly changing driving force.