Solving a homogeneous linear system $\underline{A}\,\underline{x}=\underline{0}\ ,\ \underline{x}\neq 0$? $\longleftrightarrow$ finding linear relationships among a set of vectors $\{\underline{v}_1, \underline{v}_2, \ldots\}$ (the columns of $\underline{A}$)

- yes → vectors are linearly dependent
- no → vectors are linearly independent

$\boxed{\text{INTERPRETATION}}$

3×4 system in matrix form: (3 = # eqns, 4 = # unknowns)

$$\begin{array}{c}\begin{bmatrix}1&1&3&4\\1&2&4&6\\1&3&5&8\end{bmatrix}\\ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\\ \underline{v}_1\ \underline{v}_2\ \underline{v}_3\ \underline{v}_4\end{array}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\ \xrightarrow{\text{augment}}\ \begin{bmatrix}1&1&3&4&0\\1&2&4&6&0\\1&3&5&8&0\end{bmatrix}\xrightarrow{\text{rref}}\begin{array}{c}\begin{array}{ccccc}L&L&F&F&\end{array}\\ \begin{bmatrix}1&0&2&2&0\\0&1&1&2&0\\0&0&0&0&0\end{bmatrix}\end{array}\rightarrow\begin{array}{c}\begin{array}{cccc}L_1&L_2&F_1&F_2\end{array}\\ \begin{bmatrix}1&0&2&2\\0&1&1&2\\0&0&0&0\end{bmatrix}\end{array}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

vectors in $\mathbb{R}^3$

$\boxed{\text{any relationships among these 4 vectors? central question}}$

corresponding scalar form:

$$\begin{aligned}x_1+x_2+3x_3+4x_4&=0\\x_1+2x_2+4x_3+6x_4&=0\\x_1+3x_2+5x_3+8x_4&=0\end{aligned}$$

$$\begin{aligned}x_1\quad +2x_3+2x_4&=0\\x_2+x_3+2x_4&=0\\0&=0\end{aligned}$$

corresponding vector form:

$$x_1\begin{bmatrix}1\\1\\1\end{bmatrix}+x_2\begin{bmatrix}1\\2\\3\end{bmatrix}+x_3\begin{bmatrix}3\\4\\5\end{bmatrix}+x_4\begin{bmatrix}4\\6\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\longleftrightarrow$ different vectors but same relationships (same coefficients!)

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}2\\2\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

backsub $x_3=t_1$, $x_4=t_2$ and solve:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2t_1-2t_2\\-t_1-2t_2\\t_1\\t_2\end{bmatrix}=t_1\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}+t_2\begin{bmatrix}-2\\-2\\0\\1\end{bmatrix}$$

vector coefficients of 2 independent relationships among the 4 vectors

always a 1 in free variable slots

↓

therefore you can always solve for the "free variable" columns marked "F"

$$\left.\begin{array}{l}t_1=1\\t_2=0\end{array}\right\}\ -2\begin{bmatrix}1\\1\\1\end{bmatrix}+1\begin{bmatrix}1\\2\\3\end{bmatrix}+1\begin{bmatrix}3\\4\\5\end{bmatrix}+0\begin{bmatrix}4\\6\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

(involves only $\underline{v}_1, \underline{v}_2, \underline{v}_3$)

$$\left.\begin{array}{l}t_1=0\\t_2=1\end{array}\right\}\ -2\begin{bmatrix}1\\1\\1\end{bmatrix}-2\begin{bmatrix}1\\2\\3\end{bmatrix}+0\begin{bmatrix}3\\4\\5\end{bmatrix}+1\begin{bmatrix}4\\6\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

(involves only $\underline{v}_1, \underline{v}_2, \underline{v}_4$)

$$\begin{aligned}-2\underline{v}_1-\underline{v}_2+\underline{v}_3=\underline{0}\ &\rightarrow\ \underline{v}_3=2\underline{v}_1+\underline{v}_2\\-2\underline{v}_1-2\underline{v}_2+\underline{v}_4=\underline{0}\ &\rightarrow\ \underline{v}_4=2\underline{v}_1+2\underline{v}_2\end{aligned}$$

NOTE: after reduction the free columns are easily expressed in terms of the leading columns with coefficients being their nonzero entries.

example:

$$\begin{array}{c}\begin{bmatrix}2\\1\\0\end{bmatrix}\\F_1\end{array}=2\begin{array}{c}\begin{bmatrix}1\\0\\0\end{bmatrix}\\L_1\end{array}+1\begin{array}{c}\begin{bmatrix}0\\1\\0\end{bmatrix}\\L_2\end{array}$$

corresponding original vectors have same relationship

$$\underline{v}_3=2\underline{v}_1+\underline{v}_2$$

<u>CONCLUSION</u>: all the vectors can always be expressed in terms of the "leading" columns alone.

The "leading" columns of the original set of vectors are always a linearly independent subset. [No relationships exist among them alone.]