Solving Linear Systems

$$\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}} \longleftarrow \text{equivalent systems (same solns } \vec{x}) \longrightarrow \bar{A}\vec{x} = \vec{\bar{b}}$$

new equations

Let $r = \text{rank}(A)$

augmented matrix $\underset{m\times(n+1)}{[A, \vec{b}]} \xrightarrow[\text{row reduce}]{\text{completely}} \text{rref}([A,\vec{b}]) \equiv [\bar{A}, \vec{\bar{b}}]$

$m\times(n+1)$ matrix

consistent case

$$\begin{array}{c} \\ \# \text{ independent equations in original system} \rightarrow r \le m \text{ nonzero rows} \\ \# \text{ redundant equations in original system} \rightarrow m-r \text{ zero rows} \end{array}$$

$$\begin{array}{cccccc} x_1 & & & x_n & & \\ \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & \\ 0 & 1 & \vdots & \vdots & \\ \vdots & \vdots & 0 & \vdots & \\ \vdots & \vdots & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \hdashline 0 & & & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & & & 0 & 0 \end{array}\right]$$

r leading 1's among n columns (so $r \le n$)

zero rows lead to trivial equations
$0x_1 + \cdots + 0x_n = 0$
or $0 = 0$

inconsistent case

leading 1 in last column
$[0 \cdots 0 \vdots 1]$
corresponds to equation
$0x_1 + \cdots + 0x_n = 1$
or $0 = 1$

Each of the first n columns corresponds to one of the variables $x_1, \ldots, x_n$. Label the r leading entry column variables as <u>bound</u>, the remaining $n-r$ variables as <u>free</u>.

For a given leading entry 1, in a given row, the corresponding equation can be solved for the corresponding bound variable (since it has a unit coefficient) in terms of the free variables. Since each bound variable only appears in one spot in the reduced system (zero's above and below its leading one coefficient), it does not enter into any other equation.

Set the free variables equal to arbitrary constants $t_1, t_2, \ldots t_{n-r}$ and backsubstitute these values into the equations solved for the bound variables, thus expressing *all* the variables in terms of these "parameters." *The solution is a parametrized representation of the points in the solution space of the system.*

<u>terminology</u>: different books use different names

free variables = independent variables (freely specify → parameters)

leading variables = bound variables = dependent variables (solve for)

also, changing the order of the variables changes their classification

EX. eqn for straight line: $y = mx+b$ has 2 different solns by rref technique depending on order (x,y) or (y,x)

$$mx - y = b \xrightarrow{\text{reduce}} \underset{L}{x} - \frac{\underset{F}{y}}{m} = \frac{b}{m} \xrightarrow[\text{back sub}]{} \begin{cases} x = b/m + t_1/m \\ y = t_1 \end{cases}$$

or

$$\underbrace{\underset{L}{y} - m\underset{F}{x} = b}_{\text{standard form}} \quad \text{(already reduced)} \longrightarrow \begin{cases} x = t_2 \\ y = b + mt_2 \end{cases}$$

where $t_2 = b/m + t_1/m$
or $t_1 = b + mt_2$

$\boxed{\text{Solving Linear Systems Example}}$

reduced augmented matrix for a system of 5 linear equations in 7 unknowns

$$
\begin{array}{ccccccc}
B & F & B & F & F & B & B \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7
\end{array}
$$

$$
\left[\begin{array}{ccccccc|c}
1 & -2 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 3 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{ccccccc}
 & \downarrow & & \downarrow & \downarrow & & \\
 & t_1 & & t_2 & t_3 & &
\end{array}
$$

$$
\begin{array}{lcl}
x_1 - 2x_2 + x_4 = 2 & \rightarrow & x_1 = 2x_2 - x_4 + 2 \\
x_3 + 3x_4 = 3 & \rightarrow & x_3 = -3x_4 + 3 \\
x_6 = 0 & \rightarrow & x_6 = 0 \\
x_7 = 6 & \rightarrow & x_7 = 6 \\
0 = 0 & &
\end{array}
$$

convenient equivalent system

convenient equivalent system
NOT THE SOLUTION!

$$
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 + 2 \\ x_2 \\ -3x_4 + 3 \\ x_4 \\ x_5 \\ 0 \\ 6 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 2t_1 - t_2 + 2 \\ t_1 \\ -3t_2 + 3 \\ t_2 \\ t_3 \\ 0 \\ 6 \end{bmatrix}
$$

This is the solution

all variables expressed in terms of the free variables which may take arbitrary values.
(not the solution)
(this is the result of MAPLE solve)

MAPLE
rref
backsub
(reverse order parameter indexing)

what does this mean?

any linear system of equations whose augmented matrix rrefs to the above matrix will have this solution, which can be checked by backsubstitution of the soln into those equations.

For example this system:

$$
\begin{array}{rcl}
-x_1 + 2x_2 + 4x_3 + 11x_4 - 4x_6 + x_7 &=& 16 \\
x_1 - 2x_2 + x_3 + 4x_4 - 2x_6 &=& 5 \\
-4x_3 - 12x_4 + 2x_6 + 4x_7 &=& 12 \\
-3x_1 + 6x_2 - 4x_3 - 15x_4 + 2x_6 - 4x_7 &=& -42 \\
-4x_1 + 8x_2 - x_3 - 7x_4 - x_6 + 3x_7 &=& 7
\end{array}
\longrightarrow
$$

For simplicity we only check the third equation in the system:

$$
\begin{array}{l}
-4(-3t_2+3) \stackrel{?}{=} 12 \\
-12(\ t_2\ ) \\
+2(\ 0\ ) \\
+4(\ 6\ ) \\
\hline
= 12t_2 - 12 - 12t_2 + 24 \\
= 12 \checkmark
\end{array}
$$

Only when you've checked all 5 equations can you be sure it really is a solution.