linear vocabulary

set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$:

- linear combination of — coefficients $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \xrightarrow[\text{matrix}]{\text{column}} [\underline{v_1}\,\underline{v_2}\,\underline{v_3}]\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$

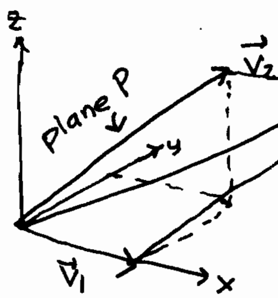
  (~~trivial~~ trivial: all coeffs zero) $0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$
- linear relationship among — $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \leftrightarrow [\underline{v_1}\,\underline{v_2}\,\underline{v_3}]\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \underline{0}$

  (nontrivial: at least one coeff nonzero) $[c_1, c_2, c_3] \neq [0,0,0]$
- linear dependent — exists nontrivial linear relationship among vectors
- linear independent — no such relationship exists
- span of — set of all possible linear combinations $\{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \mid c_1, c_2, c_3 \text{ arbitrary}\}$ = linear subspace

EXAMPLE

$$\left.\begin{aligned}\vec{v}_1 &= [1,0,0]\\ \vec{v}_2 &= [1,1,1]\\ \vec{v}_3 &= [2,1,1]\end{aligned}\right\} \text{augment}(\vec{v}_1,\vec{v}_2,\vec{v}_3) = \begin{bmatrix}1&1&2\\0&1&1\\0&1&1\end{bmatrix}$$

vectors identified as column matrices for matrix multiplication to work



note: $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

$\underline{\text{span}}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ = plane P thru origin = $\underline{\text{linear subspace}}$ of $\mathbb{R}^3$

What vectors $[x,y,z]$ lie in this plane? (ie can be expressed as a linear combination of these vectors)

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = [x,y,z]$$

$$[v_1\, v_2\, v_3]\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix} \rightarrow \left[\begin{array}{ccc|c}1&1&2&x\\0&1&1&y\\0&1&1&z\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c}1&0&1&x-y\\0&1&1&y\\0&0&0&z-y\end{array}\right]$$

consistency $z-y = 0$

answer: only vectors for which $-y + z = 0$ lie in P.
($\underline{\text{solution space}}$ of this single linear $\underline{\text{homogeneous}}$ equation in 3 unknowns x, y, z)

Are these 3 vectors linearly ind? Set $x=y=z=0$, solve for $\{c_1,c_2,c_3\}$ to find linear relationships:

$$\left[\begin{array}{ccc|c}1&1&2&0\\0&1&1&0\\0&1&1&0\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&0&0\end{array}\right] \rightarrow \begin{aligned}c_1 + c_3 &= 0\\ c_2 + c_3 &= 0\\ 0 &= 0\end{aligned}$$

soln: $c_3 = t$, $c_1 = -c_3 = -t$, $c_2 = -c_3 = -t$

$$[c_1, c_2, c_3] = [-t, -t, t] = t[-1,-1,1]$$

We found a 1-parameter family of solutions but only 1 independent solution, representing the single linear relationship among the 3 vectors: $-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$ or $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$.

Any 2 of these three vectors are linearly ind, can be taken as $\underline{\text{basis}}$ for plane.
Can always pick vectors associated with leading 1 columns, in this case

$\{\vec{v}_1, \vec{v}_2\}$ = $\underline{\text{basis}}$ of plane:
- lin ind
- any vector in plane can be expressed as a linear comb of them

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$$

c_1, c_2: "coordinates" of $\vec{v}$, vector in plane P. w.r.t. basis $\{\vec{v}_1, \vec{v}_2\}$

linear vocabulary (2)

a function f of a single variable x is like a vector with an infinite number of components:

vector v_i, $i = 1..n$ discrete index i labeling n components of $\vec{v}$

function $f(x)$, $x = a..b$ continuous index x labeling values $f(x)$ of f

linear operations:

1) addition of functions $(f_1+f_2)(x) = f_1(x) + f_2(x)$
pointwise addition

2) scalar multiplication of functions $(cf)(x) = c\,f(x)$
multiply all values by c

EXAMPLE:

$f_1(x) = 1$
$f_2(x) = x$
$f_3(x) = 1+x$ (note $f_3 = f_1+f_2$)

$\text{span}\{f_1, f_2, f_3\} = \{ c_1 \cdot 1 + c_2 x + c_3(1+x) \mid c_1, c_2, c_3 \text{ arbitrary}\}$

lin ind? $c_1 1 + c_2 x + c_3(1+x) = 0$ (zero function)

$\underbrace{(c_1+c_3)}_{=0} 1 + \underbrace{(c_2+c_3)}_{=0} x = 0$ coeffs must be zero if zero function

$c_1 + c_3 = 0$ $c_1 = -t$
$c_2 + c_3 = 0$ $c_2 = -t$
$\hookrightarrow c_3 = t$

$[c_1, c_2, c_3] = [-t, -t, t] = t[-1, -1, 1]$

one ind relationship: $-f_1 - f_2 + f_3 = 0$: $-(1) - (x) + (1+x) = 0$

or $f_3 = f_1 + f_2$: $(1+x) = (1) + (x)$

so $\text{span}\{1, x, 1+x\}$ is a 2-dimensional linear subspace with basis $\{1, x\}$ (or $\{1, 1+x\}$ or $\{x, 1+x\}$)

This is the space of linear functions of the variable x, equivalently the space of polynomials in x of degree 1 or less. $\{1, x\}$ is the natural basis like $\{(1,0),(0,1)\}$ for $\mathbb{R}^2$. The coefficients $f(x) = ax+b$ provide the corresponding natural coordinates for the 2-dimensional vector space: (a,b). These coordinates provide a natural identification of linear functions with points in the plane. Linear operations on the functions correspond to vector operations on the plane.