understanding the words of linear algebra systems talk

"system of"

3 linear eqns in 5 variables $x_1, \dots, x_5$
(scalar variables = unknowns)

$$\boxed{\begin{array}{l} 3x_1 + x_2 - 3x_3 + 11x_4 + 10x_5 = 0 \\ 5x_1 + 8x_2 + 2x_3 - 2x_4 + 7x_5 = 0 \\ 2x_1 + 5x_2 \quad\quad - x_4 + 14x_5 = 0 \end{array}}$$

single matrix equation in the vector variable $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & -4 & 14 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}\ \text{RHS matrix}}$$

"matrix form" of the linear system of eqns

$$x_1 \underbrace{\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}}_{\vec{v}_1} + x_2 \underbrace{\begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}}_{\vec{v}_2} + x_3 \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_3} + x_4 \underbrace{\begin{bmatrix} 11 \\ -2 \\ -1 \end{bmatrix}}_{\vec{v}_4} + x_5 \underbrace{\begin{bmatrix} 10 \\ 7 \\ 14 \end{bmatrix}}_{\vec{v}_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Q: can the zero vector $\vec{0}$ in $\mathbb{R}^3$ be expressed as a linear combination of $\vec{v}_1, \dots, \vec{v}_5$ in $\mathbb{R}^3$?

If so, this vector equation represents a linear relationship among them for each soln vector of coefficients

Note: Since there are at most 3 independent vectors in $\mathbb{R}^3$ there must be at least 2 independent relationships among 5 vectors

augmented matrix

$$\left[\begin{array}{ccccc|c} 3 & 1 & -3 & 11 & 10 & 0 \\ 5 & 8 & 2 & -2 & 7 & 0 \\ 2 & 5 & 0 & -4 & 14 & 0 \end{array}\right]$$

RREF, BackwardSubstitute, etc
[leading variables can be expressed in terms of free variables!]

The solution:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t_1 + 3t_2 \\ t_1 - 4t_2 \\ 2t_1 + 5t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

each value of (t_1, t_2) gives "a solution"

$$= t_1 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}_1} + t_2 \underbrace{\begin{bmatrix} 3 \\ -4 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{u}_2}$$

this represents the solution as a linear comb of 2 vectors $\vec{u}_1, \vec{u}_2$

The set of all possible solns is the "solution space".
It is the span of $\{u_1, u_2\}$
(all possible linear combinations of them)
Since $\{u_1, u_2\}$ is a set of 2 vectors which are linearly independent, they represent a <u>basis</u> of the solution space which is a linear subspace of $\mathbb{R}^5$
= 2-plane thru origin of $\mathbb{R}^5$

all possible linear relationships among $\vec{v}_1, \dots, \vec{v}_5$: $(-2t_1 + 3t_2)\vec{v}_1 + (t_1 - 4t_2)\vec{v}_2 + (2t_1 + 5t_2)\vec{v}_3 + t_1\vec{v}_4 + t_2\vec{v}_5 = \vec{0}$

2 independent such relationships correspond to $\vec{u}_1, \vec{u}_2$:

$$\left\{ \begin{array}{l} -2v_1 + v_2 + 2v_3 + v_4 = 0 \\ 3v_1 - 4v_2 + 5v_3 + v_5 = 0 \end{array} \right.$$

(can be used to express $\vec{v}_4, \vec{v}_5$ in terms of $\vec{v}_1, \vec{v}_2, \vec{v}_3$)

[free columns can be expressed in terms of the leading columns!]

Note that setting $x_5 = -1$ leads to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 - \vec{v}_5 = 0$
or $\underline{\vec{v}_5 = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4}$

solving this linear system for $x_1, \dots, x_4$ has same augmented matrix without final zero column.