

linear combinations forwards and backwards

- "Take a linear combination" of $\{\langle 1,2,3 \rangle, \langle 2,3,1 \rangle, \langle 3,1,2 \rangle\}$ with coefficients 2, 1, -1:

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+2-3 \\ 4+3-1 \\ 6+1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix} \text{ done.}$$

- Now undo this operation:

"Express $\langle 1,6,5 \rangle$ as a linear combination" of $\{\langle 1,2,3 \rangle, \langle 2,3,1 \rangle, \langle 3,1,2 \rangle\}$:

so we must find coefficients c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}}_{\text{matrix form of linear system of equations for unknowns } c_1, c_2, c_3} \xrightarrow{\text{augment}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 6 \\ 3 & 1 & 2 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{\text{back sub}} \xrightarrow{\text{sub}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ not done yet.}$$

matrix form of linear system of equations for unknowns c_1, c_2, c_3

express $\langle 1,6,5 \rangle$ as a linear combination:

$$\begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \text{ done.}$$

- Now express any vector $\langle x_1, x_2, x_3 \rangle$ in terms of these three vectors.

Note that $B = \text{augment}(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ row reduced to the identity

and $\det B = -18 \neq 0$, either fact of which means they are linearly independent and hence form a basis of \mathbb{R}^3 .

$$y_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow[\text{equivalent to}]{\text{solve as above}} \vec{y} = B^{-1} \vec{x} \quad B^{-1} = \frac{1}{-18} \begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$$

$$B \vec{y} = \vec{x}$$



old
coords

$$\begin{aligned} x_1 &= y_1 + 2y_2 + 3y_3 \\ x_2 &= 2y_1 + 3y_2 + y_3 \\ x_3 &= 3y_1 + y_2 + 2y_3 \end{aligned}$$

linear
change
of
coordinates
on \mathbb{R}^3

$$\begin{aligned} y_1 &= \frac{1}{-18}(-5x_1 + x_2 + 7x_3) \\ y_2 &= \frac{1}{-18}(x_1 + 7x_2 - 5x_3) \\ y_3 &= \frac{1}{-18}(7x_1 - 5x_2 + x_3) \end{aligned}$$

new
coords

old basis:

$$\{\langle 1,0,0 \rangle, \langle 0,1,0 \rangle, \langle 0,0,1 \rangle\}$$

new basis:

$$\{\langle 1,2,3 \rangle, \langle 2,3,1 \rangle, \langle 3,1,2 \rangle\}$$

$$\vec{x} = x_1 \langle 1,0,0 \rangle + x_2 \langle 0,1,0 \rangle + x_3 \langle 0,0,1 \rangle = y_1 \langle 1,2,3 \rangle + y_2 \langle 2,3,1 \rangle + y_3 \langle 3,1,2 \rangle$$