leading and free variables and columns

$$A_{m \times n} = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} = [C_1 \cdots C_n]$$

A matrix can be looked at as an ordered set of rows or as an ordered set of columns.

$$\boxed{A\vec{x} = \vec{0}} \iff \begin{bmatrix} \vec{R_1} \cdot \vec{x} \\ \vdots \\ \vec{R_m} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

two complementary views of the same matrix eqn!

The rows here are the coefficient vectors of a set of linear relationships satisfied by the scalar variables.

The reduced row echelon form of these equations allows the leading scalar variables to be expressed as linear functions of the free variables which in turn can take any real values, represented by assigning them to arbitrary constants (parameters).

$$x_1 \vec{C_1} + \cdots + x_n \vec{C_n} = \vec{0}$$

This represents a linear relationship among the columns if the coefficient vector $\vec{x} \neq \vec{0}$. This relationship only involves those columns whose coefficient in this relationship is nonzero.

The solution of the linear system of equations represents the most general coefficient vector associated with linear relationships among the columns (excluding the zero soln!).

It consists of an arbitrary linear combination of a set of coefficient vectors which is automatically a linearly independent set since if $\vec{x} = t_1 \vec{v_1} + \cdots + t_p \vec{v_p} = \vec{0}$ the free variable parameters $t_1, \ldots, t_p$ are forced to be zero.

Each coefficient vector $\vec{v_i}$ represents a linear relationship which allows one free column to be expressed as a linear combination of the leading columns.

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complementarity:

the leading scalar variables can be expressed as linear combinations of the free variables

BUT

the free columns can be expressed as linear combinations of the leading columns.

leading & free are exchanged going from variables to columns.

The reduced row echelon form algorithm allows us to see both of these facts by inspection (just looking at the reduced matrix and its equivalent system of equations).

The free variables can be considered as independent, while the set of leading columns is linearly independent.