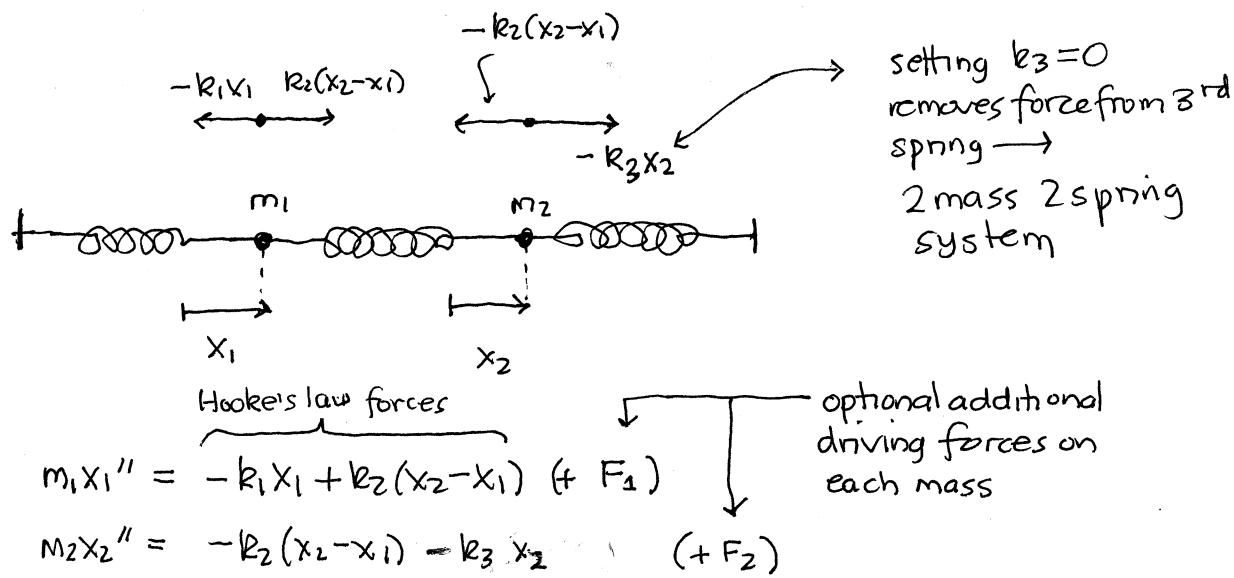


2 mass 3 spring system

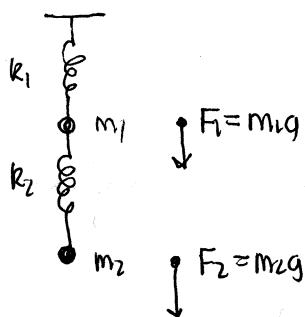


$$\ddot{x}_1 = -\frac{(k_1+k_2)}{m_1} x_1 + \frac{k_2}{m_1} x_2 \quad (+ \frac{F_1}{m_1})$$

$$\ddot{x}_2 = \frac{k_2}{m_2} x_1 - \frac{(k_2+k_3)}{m_2} x_2 \quad (+ \frac{F_2}{m_2})$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -(k_2+k_3)/m_2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_X + \underbrace{\begin{bmatrix} F_1/m_1 \\ F_2/m_2 \end{bmatrix}}_f$$

2 mass 2 spring system with gravity (set k3=0)



Remove spring 3, hung from ceiling, constant gravitational force pulls down, look for equilibrium solns $\vec{x} = \vec{x}_0$ (constant)

$$\vec{x}'' = A\vec{x} + \vec{f}$$

$$0 = \vec{x}_0'' = A\vec{x}_0 + \vec{f} \rightarrow \text{solve } \vec{x}_0 = -A^{-1}\vec{f} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

equilibrium positions.

$$\text{Note } (\vec{x} - \vec{x}_0)'' = \vec{x}'' = A\vec{x} + \vec{f} = A(\vec{x} - \vec{x}_0 + \vec{x}_0) + \vec{f}$$

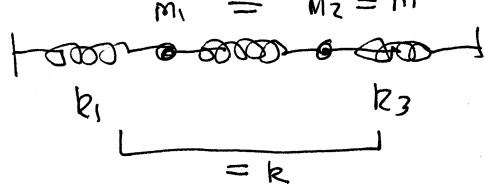
$$= A(\vec{x} - \vec{x}_0) + \underbrace{A\vec{x}_0 + \vec{f}}_0 = 0 \text{ for equilibrium}$$

$$\text{so } (\vec{x} - \vec{x}_0)'' = A(\vec{x} - \vec{x}_0)$$

Departures from equilibrium satisfy the homogeneous equations which are oscillations.

2 mass 3 spring system (2)

special case



symmetric about center → expects solns to reflect this symmetry

$$A = \begin{bmatrix} -\frac{(k_1+k_2)}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{(k_1+k_2)}{m} \end{bmatrix}$$

$$\frac{k_1}{m} = 1 \quad \frac{k_2}{m} = \frac{3}{2}$$

$$= \begin{bmatrix} -5/2 & 3/2 \\ 3/2 & -5/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}$$

from HW previously done.
now can interpret soln

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{A_1 \cos(t-\delta_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{slow mode } \omega=1} + \underbrace{A_2 \cos(2t-\delta_2) \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{fast mode } \omega=2}$$

homogeneous general soln
(no driving forces)

equal amplitudes,
same direction

equal amplitudes,
opposite direction

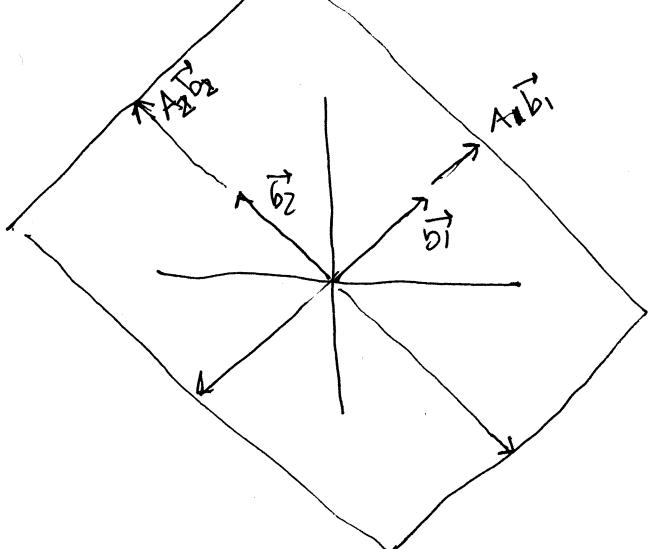
$$\begin{array}{c} \xrightarrow{x_1} \quad \xleftarrow{x_2} \\ \xrightarrow{x_1} \quad \xleftarrow{x_2} \end{array}$$

$\frac{x_1}{x_2} = 1$ at all times
" tandem mode."

$$\begin{array}{c} \xleftarrow{x_1} \quad \xrightarrow{x_2} \\ \xleftarrow{x_1} \quad \xleftarrow{x_2} \end{array}$$

$\frac{x_1}{x_2} = -1$ at all times
" accordian mode"

can explicitly write down soln for all 5 parameters k_1, k_2, k_3, m_1, m_2 using quad formula for eigenvalues, backsub, either row of matrix gives ratio x_1/x_2 for eigenvector. [all have slow tandem mode, fast accordian mode]



For given initial data leading to A_1, A_2 amplitudes, solution curve confined to rectangle. hits edge when one oscillation is at its extreme value.

$$\vec{x} = \underbrace{\cos(\omega t - \delta_1) (A_1 \vec{b}_1)}_{= \pm 1 \text{ hits edge}} + \underbrace{\cos(\omega_2 t - \delta_2) (A_2 \vec{b}_2)}_{= \pm 1 \text{ hits edge}}$$

$$\vec{x}'' = A\vec{x} \quad \text{IVP (2x2 coefficient matrix)}$$

$$\left. \begin{array}{l} x_1'' = (-5x_1 + 3x_2)/2, \quad x_1(0) = 1, \quad x_1'(0) = 0 \\ x_2'' = (3x_1 - 5x_2)/2, \quad x_2(0) = 0, \quad x_2'(0) = 1 \end{array} \right\} \text{IVP}$$

$$\left[\begin{array}{l} x_1 \\ x_2 \end{array} \right]'' = \underbrace{\frac{1}{2} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}}_A \left[\begin{array}{l} x_1 \\ x_2 \end{array} \right], \quad \left[\begin{array}{l} x_1(0) \\ x_2(0) \end{array} \right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \left[\begin{array}{l} x_1'(0) \\ x_2'(0) \end{array} \right] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{matrix form}$$

$$0 = \det(A - \lambda I) = \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) \quad \left. \begin{array}{l} \text{diagonalization,} \\ \lambda = -1, -4 \rightarrow \dots \quad B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{array} \right\} \text{new variables}$$

$$B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\vec{x} = B\vec{y} = y_1 \vec{b}_1 + y_2 \vec{b}_2, \quad \vec{y} = B^{-1}\vec{x}$$

$$\vec{x}'' = A\vec{x} \rightarrow B^{-1}(B\vec{y})'' = BAB\vec{y}$$

$$\vec{y}'' = A_B \vec{y}$$

$$\left[\begin{array}{l} y_1 \\ y_2 \end{array} \right]'' = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \left[\begin{array}{l} y_1 \\ y_2 \end{array} \right] = \begin{bmatrix} -y_1 \\ -4y_2 \end{bmatrix}$$

$$\begin{aligned} y_1'' &= -y_1 & y_1 &= C_1 \cos t + C_2 \sin t \\ y_2'' &= -4y_2 & y_2 &= C_3 \cos 2t + C_4 \sin 2t \end{aligned}$$

$$\left[\begin{array}{l} x_1 \\ x_2 \end{array} \right] = B \begin{bmatrix} C_1 \cos t + C_2 \sin t \\ C_3 \cos 2t + C_4 \sin 2t \end{bmatrix} \quad \downarrow \text{solving ICs}$$

$$\left[\begin{array}{l} x_1' \\ x_2' \end{array} \right] = B \begin{bmatrix} -C_1 \sin t + C_2 \cos t \\ -2C_3 \sin 2t + 2C_4 \cos 2t \end{bmatrix}$$

$$\left[\begin{array}{l} 1 \\ 0 \end{array} \right] = \left[\begin{array}{l} x_1(0) \\ x_2(0) \end{array} \right] = B \begin{bmatrix} C_1 \\ C_3 \end{bmatrix}, \quad \left[\begin{array}{l} C_1 \\ C_3 \end{array} \right] = B^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{l} 0 \\ 1 \end{array} \right] = \left[\begin{array}{l} x_1'(0) \\ x_2'(0) \end{array} \right] = B \begin{bmatrix} C_2 \\ 2C_4 \end{bmatrix}, \quad \left[\begin{array}{l} C_2 \\ 2C_4 \end{array} \right] = B^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{l} C_2 \\ C_4 \end{array} \right] = \left[\begin{array}{l} 1/2 \\ 1/4 \end{array} \right]$$

$$\left[\begin{array}{l} y_1 \\ y_2 \end{array} \right] = \left[\begin{array}{l} \frac{1}{2}(\cos t + \sin t) \\ \frac{1}{4}(-2\cos 2t + \sin 2t) \end{array} \right] = \left[\begin{array}{l} \frac{1}{2} \cos(t - \pi/4) \\ \frac{1}{4} \cos(2t - (\pi - \arctan 1/2)) \end{array} \right]$$

$$\left[\begin{array}{l} x_1 \\ x_2 \end{array} \right] = \left[\begin{array}{l} 1 - 1 \\ 1 - 1 \end{array} \right] \left[\begin{array}{l} (\cos t + \sin t)/2 \\ (-2\cos 2t + \sin 2t)/4 \end{array} \right]$$

$$= \left[\begin{array}{l} \frac{1}{2} \cos t + \frac{1}{2} \sin t + \frac{1}{2} \cos 2t - \frac{1}{4} \sin 2t \\ \frac{1}{2} \cos t - \frac{1}{2} \sin t - \frac{1}{2} \cos 2t + \frac{1}{4} \sin 2t \end{array} \right]$$

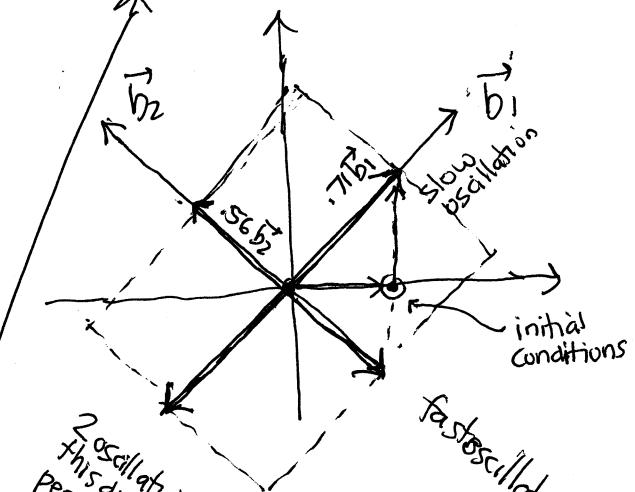
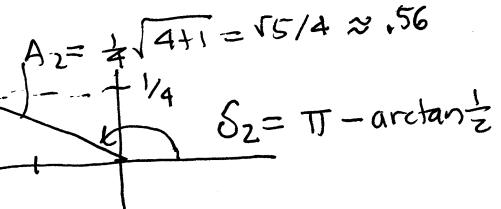
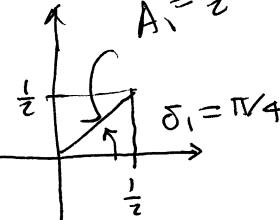
$$\vec{x} = \frac{1}{2} \cos(t - \delta_1) \vec{b}_1 + \frac{\sqrt{5}}{4} \cos(2t - \delta_2) \vec{b}_2$$

slow mode

fast mode

phase-shifted cosine calculation

$$A_1 = \frac{1}{2} \sqrt{2} = \frac{1}{\sqrt{2}} \approx .71$$



oscillations
in this direction
per 1 oscillation
in the other so we have to get a figure 8 like curve

figure 8 like curve

figure 8 curve.mw

driven case

$$\begin{aligned} & + \begin{bmatrix} 0 \\ 50 \cos 3t \end{bmatrix} + 50 \cos 3t \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = & \underbrace{\begin{bmatrix} -5/2 & 3/2 \\ 3/2 & -5/2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5/2 x_1 + 3/2 x_2 \\ 3/2 x_1 - 5/2 x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & \lambda = -1 \pm 4 \end{aligned}$$

decoupling
soln:

$$\vec{x} = B \vec{y} = y_1 \vec{b}_1 + y_2 \vec{b}_2$$

$$\vec{x}'' = A \vec{x} + \vec{F} \rightarrow \vec{y}'' = A_B \vec{y} + B^{-1} \vec{F}: \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 25 \cos 3t \\ 25 \cos 3t \end{bmatrix} = \begin{bmatrix} -y_1 + 25 \cos 3t \\ -4y_2 + 25 \cos 3t \end{bmatrix}$$

$$\begin{cases} y_1'' + y_1 = 25 \cos 3t \\ y_2'' + 4y_2 = 25 \cos 3t \end{cases}$$

$$y_{1h} = C_1 \cos 3t + C_2 \sin 3t$$

$$y_{2h} = C_3 \cos 2t + C_4 \sin 2t$$

$$y_{1p} = C_5 \cos 3t + C_6 \sin 3t$$

$$y_{2p} = C_7 \cos 3t + C_8 \sin 3t$$

$$y_{1p}'' + y_{1p} = -9C_5 \cos 3t - 9C_6 \sin 3t + 6 \cos 3t + C_6 \sin 3t = \frac{-8C_5}{25} \cos 3t - \frac{8C_6}{25} \sin 3t = 25 \cos 3t$$

$$C_5 = -\frac{25}{8}, \quad C_6 = 0.$$

$$y_{2p}'' + 4y_{2p} = -9C_7 \cos 3t - 9C_8 \sin 3t + 4C_7 \cos 3t + 4C_8 \sin 3t = \frac{-5C_7}{25} \cos 3t - \frac{5C_8}{25} \sin 3t = 25 \cos 3t$$

$$C_7 = -\frac{25}{5} = -5, \quad C_8 = 0$$

initial values

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \cos 3t + C_2 \sin 3t \\ C_3 \cos 2t + C_4 \sin 2t \end{bmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{general soln}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -C_1 \sin 3t + C_2 \cos 3t \\ -2C_3 \sin 2t + 2C_4 \cos 2t \end{bmatrix} + \begin{bmatrix} 75/8 \sin 3t \\ 15 \sin 3t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 - 25/8 \\ C_3 - 5 \end{bmatrix} \quad \begin{bmatrix} C_1 - 25/8 \\ C_3 - 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \quad \begin{array}{l} C_1 = 1/2 + 25/8 = 29/8 \\ C_3 = 5 - 1/2 = 9/2 \end{array}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_2 \\ 2C_4 \end{bmatrix} \quad \begin{bmatrix} C_2 \\ 2C_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \rightarrow \begin{array}{l} C_2 = 1/2 \\ C_4 = 1/4 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{29}{8} \cos 3t + \frac{1}{2} \sin 3t - \frac{25}{8} \cos 3t \\ \frac{9}{2} \cos 2t + \frac{1}{4} \sin 2t - 5 \cos 3t \end{bmatrix} \quad \leftarrow \text{matrix form}$$

$$= \begin{bmatrix} \frac{29}{8} \cos 3t + \frac{1}{2} \sin 3t - \frac{25}{8} \cos 3t & -\frac{9}{2} \cos 2t - \frac{1}{4} \sin 2t + 5 \cos 3t \\ \frac{29}{8} \cos 3t + \frac{1}{2} \sin 3t - \frac{25}{8} \cos 3t & + \frac{9}{2} \cos 2t + \frac{1}{4} \sin 2t - 5 \cos 3t \end{bmatrix}$$

$$= \left(\frac{29}{8} \cos 3t + \frac{1}{2} \sin 3t \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{9}{2} \cos 2t + \frac{1}{4} \sin 2t \right) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{8} \cos 3t \begin{bmatrix} 3 \\ 1/3 \end{bmatrix} \quad \leftarrow \text{vector linear comb form}$$

$\omega = 1$ mode

$\omega = 2$ mode

$\omega = 3$ response mode

$\cos(t - \delta)$

$$\delta_1 = \arctan\left(\frac{4}{29}\right) \approx 3.9^\circ$$

$$A_1 = \frac{1}{2} \sqrt{29^2 + 9^2} \approx 3.46$$

$$\delta_2 = \arctan\left(\frac{1}{4}\right) \approx 0.009^\circ$$

$$A_2 = \frac{1}{2} \sqrt{18^2 + 1^2} \approx 4.91$$

(see Maple worksheet graphic)