$\vec{x}'' = A\vec{x}$ IVP (2x2 coefficient matrix)

$$\left.\begin{aligned} x_1'' &= (-5x_1+3x_2)/2, \quad x_1(0)=1, \quad x_1'(0)=0 \\ x_2'' &= (3x_1-5x_2)/2, \quad x_2(0)=0, \quad x_2'(0)=1 \end{aligned}\right\} \text{IVP}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\frac{1}{2}\begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{matrix form}$$

$$\left.\begin{aligned} &0 = \det(A-\lambda I) = \lambda^2+5\lambda+4 = (\lambda+1)(\lambda+4) \\ &\lambda = -1,-4 \quad \to \cdots \quad B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \\ &\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2, \quad \vec{y} = B^{-1}\vec{x} \end{aligned}\right\} \text{diagonalization, new variables}$$

$$\left.\begin{aligned} &\vec{x}'' = A\vec{x} \longrightarrow B^{-1}(B\vec{y})' = BAB\vec{y} \\ &\vec{y}'' = A_B\vec{y} \\ &\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_1 \\ -4y_2 \end{bmatrix} \\ &y_1'' = -y_1 \qquad y_1 = c_1\cos t + c_2 \sin t \\ &y_2'' = -4y_2 \qquad y_2 = c_3\cos 2t + c_4 \sin 2t \end{aligned}\right\} \text{decoupling DEs, solving them}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\begin{bmatrix} c_1\cos t + c_2\sin t \\ c_3\cos 2t + c_4 \sin 2t \end{bmatrix}$$

$\downarrow$ solving ICs

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = B\begin{bmatrix} -c_1\sin t + c_2\cos t \\ -2c_3\sin 2t + 2c_4\cos 2t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 \\ c_3 \end{bmatrix}; \quad \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = B\begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix}, \quad \begin{pmatrix} c_2 \\ 2c_4 \end{pmatrix} = B^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} c_2 \\ c_4 \end{pmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\cos t + \sin t) \\ \frac{1}{4}(-2\cos 2t + \sin 2t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos(t - \pi/4) \\ \frac{\sqrt{5}}{4}\cos(2t - (\pi - \arctan 1/2)) \end{bmatrix}$$

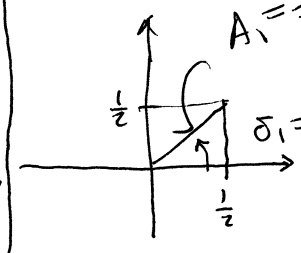
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (\cos t + \sin t)/2 \\ (-2\cos 2t + \sin 2t)/4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t - \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t \end{bmatrix}$$

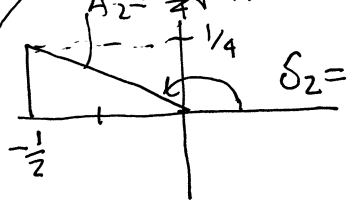
$$\vec{x} = \underbrace{\frac{1}{\sqrt{2}}\cos(t-\delta_1)\,\vec{b}_1}_{\text{slow mode}} + \underbrace{\frac{\sqrt{5}}{4}\cos(2t-\delta_2)\,\vec{b}_2}_{\text{fast mode}}$$

phase-shifted cosine calculation

$A_1 = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}} \approx .71$



$A_2 = \frac{1}{4}\sqrt{4+1} = \sqrt{5}/4 \approx .56$



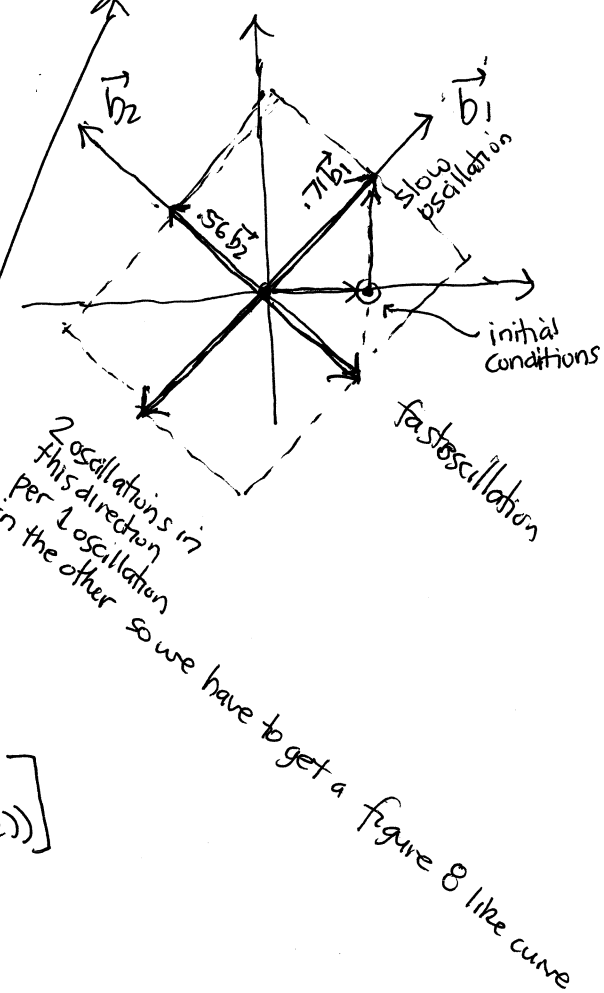


figure8curve.mw

driven case

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -5/2 & 3/2 \\ 3/2 & -5/2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 50\cos 3t \end{bmatrix} = \begin{bmatrix} -5/2\, x_1 + 3/2\, x_2 \\ 3/2\, x_1 - 5/2\, x_2 + 50\cos 3t \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\boxed{\text{decoupling soln:}}$

$\lambda = -1, -4$

$$A \rightarrow B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$B^{-1}\vec{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 50\cos 3t \end{bmatrix} = \begin{bmatrix} 25\cos 3t \\ 25\cos 3t \end{bmatrix}$$

$$\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2$$

$$\vec{x}'' = A\vec{x} + \vec{F} \rightarrow \vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}: \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 25\cos 3t \\ 25\cos 3t \end{bmatrix} = \begin{bmatrix} -y_1 + 25\cos 3t \\ -4y_2 + 25\cos 3t \end{bmatrix}$$

$$\boxed{\begin{aligned} y_1'' + y_1 &= 25\cos 3t \\ y_2'' + 4y_2 &= 25\cos 3t \end{aligned}} \quad \begin{aligned} y_{1h} &= c_1\cos t + c_2\sin t & y_{1p} &= c_5\cos 3t + c_6\sin 3t \\ y_{2h} &= c_3\cos 2t + c_4\sin 2t & y_{2p} &= c_7\cos 3t + c_8\sin 3t \end{aligned}$$

$$y_{1p}'' + y_{1p} = -9c_5\cos 3t - 9c_6\sin 3t + c_5\cos 3t + c_6\sin 3t = \underbrace{-8c_5}_{=25}\cos 3t \underbrace{-8c_6}_{=0}\sin 3t = 25\cos 3t$$

$$c_5 = -\frac{25}{8} \qquad c_6 = 0.$$

$$y_{2p}'' + 4y_{2p} = -9c_7\cos 3t - 9c_8\sin 3t + 4c_7\cos 3t + 4c_8\sin 3t = \underbrace{-5c_7}_{=25}\cos 3t \underbrace{-5c_8}_{=0}\sin 3t = 25\cos 3t$$

$$c_7 = -\frac{25}{5} = -5 \qquad c_8 = 0$$

initial values

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\cos t + c_2\sin t - \frac{25}{8}\cos 3t \\ c_3\cos 2t + c_4\sin 2t - 5\cos 3t \end{bmatrix} \quad \Big\} \text{ general soln}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -c_1\sin t + c_2\cos t + \frac{75}{8}\sin 3t \\ -2c_3\sin 2t + 2c_4\cos 2t + 15\sin 3t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 - 25/8 \\ c_3 - 5 \end{bmatrix} \quad \begin{bmatrix} c_1 - 25/8 \\ c_3 - 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \quad \begin{aligned} c_1 &= 1/2 + 25/8 = 29/8 \\ c_3 &= 5 - 1/2 = 9/2 \end{aligned}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix} \quad \begin{bmatrix} c_2 \\ 2c_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \rightarrow \begin{aligned} c_2 &= 1/2 \\ c_4 &= 1/4. \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{29}{8}\cos t + \frac{1}{2}\sin t - \frac{25}{8}\cos 3t \\ \frac{9}{2}\cos 2t + \frac{1}{4}\sin 2t - 5\cos 3t \end{bmatrix} \quad \leftarrow \text{matrix form}$$

$$= \begin{bmatrix} \frac{29}{8}\cos t + \frac{1}{2}\sin t - \frac{25}{8}\cos 3t - \frac{9}{2}\cos 2t - \frac{1}{4}\sin 2t + 5\cos 3t \\ \frac{29}{8}\cos t + \frac{1}{2}\sin t - \frac{25}{8}\cos 3t + \frac{9}{2}\cos 2t + \frac{1}{4}\sin 2t - 5\cos 3t \end{bmatrix} \quad \leftarrow \text{scalar form (top, bottom)}$$

$\frac{15}{8}$ $\quad -\frac{65}{8}$

$$= \left(\frac{29}{8}\cos t + \frac{1}{2}\sin t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{9}{2}\cos 2t + \frac{1}{4}\sin 2t\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{8}\cos 3t\begin{bmatrix} 3 \\ -13 \end{bmatrix} \quad \leftarrow \text{vector linear comb form}$$

$\omega = 1$ natural mode $\qquad \omega = 2$ natural mode $\qquad \omega = 3$ response mode

$\cos(t - \delta_1)$

$\delta_1 = \arctan\left(\frac{4}{29}\right) \approx 3.9°$

$A_1 = \frac{1}{8}\sqrt{29^2 + 4^2} \approx 3.66$

$\delta_2 = \arctan\left(\frac{1}{18}\right) \approx 0.009°$

$A_2 = \frac{1}{4}\sqrt{18^2 + 1^2} \approx 4.51$

(see Maple worksheet graphic)