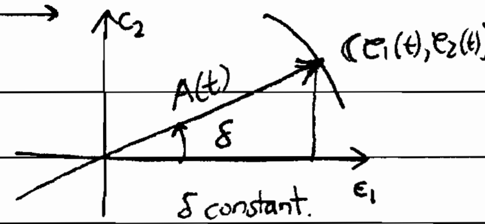
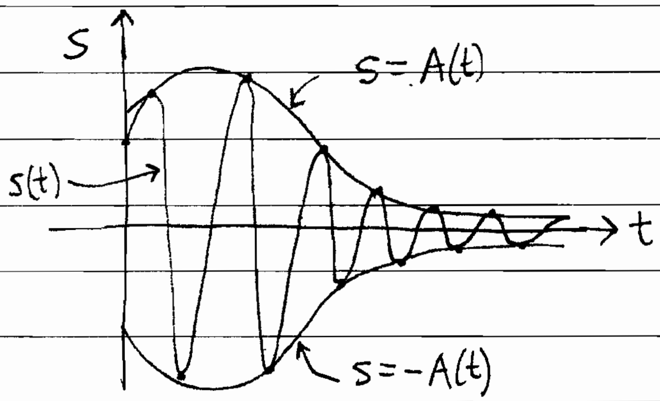
EXPONENTIALLY MODULATED SINUSOIDAL FUNCTIONS

amplitude modulation

variable coefficients

$$s(t) = \underbrace{A(t)}_{\substack{\text{variable}\\ \text{amplitude}\\ \geq 0}} \cos(\underbrace{\omega t - \delta}_{\text{constants}}) = \underbrace{c_1(t)}_{A(t)\cos\delta} \cos\omega t + \underbrace{c_2(t)}_{A(t)\sin\delta} \sin\omega t$$





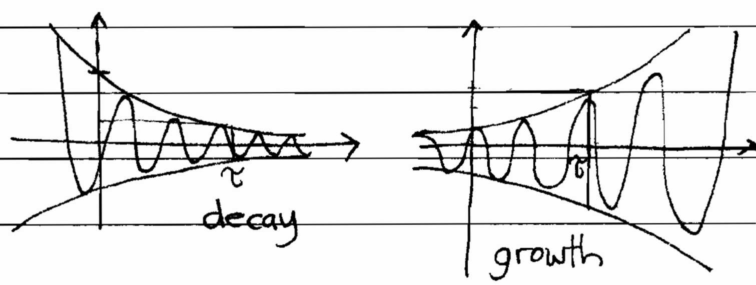
For t values where $\cos(\omega t - \delta) = \pm 1$, the graph of $s(t)$ touches the graph of $\pm A(t)$ respectively, while for t values where $\cos(\omega t - \delta) = 0$ the graph of $s(t)$ crosses the t axis.

The graphs $s = \pm A(t)$ define the "envelope" of the signal, which oscillates between them at fixed frequency ω.

exponential envelope: $A(t) = A_0 e^{\pm t/\tau}$, $\tau > 0$ $\begin{cases} + & \text{growth} \\ - & \text{decay} \end{cases}$

A_0 = initial amplitude $= A(0)$

τ = characteristic constant (time) for exponential behavior ("e-folding" time)



Then

(I) $s(t) = A_0 e^{\pm t/\tau} \cos(\omega t - \delta)$

$= e^{\pm t/\tau} \left[c_1 \cos\omega t + c_2 \sin\omega t \right]$ where $(c_1, c_2) = A_0(\cos\delta, \sin\delta)$ are related in the usual way for the sinusoidal signal in square brackets

optional/ignore:

Complex form (electrical engineering/physics)

Let $\mathbf{C} = c_1 + i c_2 = A_0 e^{i\delta}$. Then

$$s(t) = \underbrace{A_0 e^{\pm t/\tau}}_{\text{real}} \operatorname{Re}\left[e^{i(\omega t - \delta)}\right] = \operatorname{Re}\left[A_0 e^{\pm t/\tau} e^{i(\omega t - \delta)}\right]$$

$$= \operatorname{Re}\Big[\underbrace{A_0 e^{-i\delta}}_{\bar{\mathbf{C}}} \, e^{(\pm 1/\tau + i\omega)t}\Big]$$

positive Re part $\leftrightarrow$ growth → Real part is coefficient of exponential argument

negative Re part $\leftrightarrow$ decay

Im part is coefficient of trig argument

define $iW = i\omega \pm 1/\tau = i(\omega \mp i/\tau)$ $= a + ib$ (book)

or $W = \omega \mp i/\tau$ "complex frequency"

negative imag. part $\leftrightarrow$ growth

positive imag. part $\leftrightarrow$ decay

(II) $= \operatorname{Re}\left[\bar{\mathbf{C}} e^{iWt}\right]$

HW Problem: $s(t) = e^{-2t}(-3\cos 5t + 4\sin 5t)$.

Find $\omega, \tau, W, c_1, c_2, A_0, \delta, \mathbf{C}$. Express $s(t)$ in form (I) and (II).

~~Show that this solves the IVP: $s'' + 4s' + 29s = 0$, $s(0) = -3$, $s'(0) = 26$.~~