E&P 7.3.37 (see below right corner first)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \underbrace{\begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 55 \\ 0 \\ 0 \end{bmatrix}$$

eigenvalues & eigenvectors

$$|A - \lambda I| = -(\lambda^3 + 6\lambda^2 + 11\lambda) = -\lambda(\lambda^2 + 6\lambda + 11) = 0$$

$$\lambda = 0, -3+\sqrt{2}i, -3-\sqrt{2}i \quad \text{(quad formula)}$$

$$\lambda = 0: \ A - \lambda I = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{ccc} L & L & F \end{array} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 2t$, $x_2 = 2/3\,t$, $x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \underbrace{\begin{bmatrix} 2 \\ 2/3 \\ 1 \end{bmatrix}}_{\vec{b}_1}$$

$$\lambda = -3+\sqrt{2}i: \ A - \lambda I = \begin{bmatrix} -1-(-3+\sqrt{2}i) & 0 & 2 \\ 1 & -3-(-3+\sqrt{2}i) & 0 \\ 0 & 3 & -2-(-3+\sqrt{2}i) \end{bmatrix} = \begin{bmatrix} 2-\sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ 0 & 3 & 1-\sqrt{2}i \end{bmatrix}.$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{-2}{-2+\sqrt{2}i} \\ 0 & 1 & \frac{i\sqrt{2}}{-2+\sqrt{2}i} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{alize}]{\text{ration}} \begin{array}{ccc} L & L & F \end{array} \begin{bmatrix} 1 & 0 & (2+\sqrt{2}i)/3 \\ 0 & 1 & (1-\sqrt{2}i)/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -\frac{(2+\sqrt{2}i)}{3}t$, $x_2 = \frac{(-1+\sqrt{2}i)}{3}t$, $x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \underbrace{\begin{bmatrix} -(2+\sqrt{2}i)/3 \\ (-1+\sqrt{2}i)/3 \\ 1 \end{bmatrix}}_{\vec{b}_2}$$

$$\lambda = -3-\sqrt{2}i: \rightarrow \vec{b}_3 = \overline{\vec{b}_2} = \begin{bmatrix} -(2-\sqrt{2}i)/3 \\ (-1-\sqrt{2}i)/3 \\ 1 \end{bmatrix}$$

complex conjugates!

Result: $\lambda = 0, \ -3+\sqrt{2}i, \ -3-\sqrt{2}i$

$$B = \begin{bmatrix} 2 & -(2+\sqrt{2}i)/3 & -(2-\sqrt{2}i)/3 \\ 2/3 & (-1+\sqrt{2}i)/3 & (-1-\sqrt{2}i)/3 \\ 1 & 1 & 1 \end{bmatrix}$$

Transform DE system:

$$\vec{x} = B\vec{y}, \quad \vec{y} = B^{-1}\vec{x}, \quad \vec{y}' = A_B\vec{y} \rightarrow$$

$$A_B = B^{-1}AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3+\sqrt{2}i & 0 \\ 0 & 0 & -3-\sqrt{2}i \end{bmatrix}$$

diagonal! →

decoupled equations:

$y_1' = 0 \rightarrow y_1 = c_1$

$y_2' = (-3+\sqrt{2}i)y_2 \rightarrow y_2 = c_2 e^{(-3+\sqrt{2}i)t}$

$y_3' = (-3-\sqrt{2}i)y_3 \rightarrow y_3 = c_3 e^{(-3-\sqrt{2}i)t} \longrightarrow c_3 = \overline{c_2}$ for real solns

$$\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2 + y_3\vec{b}_3 = c_1 \begin{bmatrix} 2 \\ 2/3 \\ 1 \end{bmatrix} + c_2 e^{(-3+\sqrt{2}i)t}\vec{b}_2 + \text{c.c.}$$

complex conjugate

where does i go?!

need real lin. comb of Re/Im parts of circled vector

side calculation:

$$e^{(-3+\sqrt{2}i)t}\vec{b}_2 = e^{-3t}(\cos\sqrt{2}t + i\sin\sqrt{2}t) \begin{bmatrix} -(2+\sqrt{2}i)/3 \\ (-1+\sqrt{2}i)/3 \\ 1 \end{bmatrix} = e^{-3t} \begin{bmatrix} (-2\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t)/3 + i(-2\sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t)/3 \\ (-\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t)/3 + i(-\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t)/3 \\ \cos\sqrt{2}t + i\sin\sqrt{2}t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2/3 \\ 1 \end{bmatrix} + c_2 e^{-3t} \overbrace{\begin{bmatrix} (-2\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t)/3 \\ (-\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t)/3 \\ \cos\sqrt{2}t \end{bmatrix}}^{\text{Re}(\ )} + c_3 e^{-3t} \overbrace{\begin{bmatrix} (-2\sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t)/3 \\ (-\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t)/3 \\ \sin\sqrt{2}t \end{bmatrix}}^{\text{Im}(\ )}$$

general solution

$$\begin{bmatrix} 55 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2/3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -\sqrt{2}/3 \\ \sqrt{2}/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -2/3 & -\sqrt{2}/3 \\ 2/3 & -1/3 & \sqrt{2}/3 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -15 \\ -45\sqrt{2}/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 15 \end{bmatrix} + -15e^{-3t} \begin{bmatrix} (-2\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t)/3 \\ (-\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t)/3 \\ \cos\sqrt{2}t \end{bmatrix} - \frac{45\sqrt{2}}{2}e^{-3t} \begin{bmatrix} (-2\sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t)/3 \\ (-\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t)/3 \\ \sin\sqrt{2}t \end{bmatrix}$$

$$= \begin{bmatrix} 30 + 25e^{-3t}\cos\sqrt{2}t + 10\sqrt{2}e^{-3t}\sin\sqrt{2}t \\ 10 - 10e^{-3t}\cos\sqrt{2}t + \frac{25}{2}\sqrt{2}e^{-3t}\sin\sqrt{2}t \\ 15 - 15e^{-3t}\cos\sqrt{2}t - \frac{45}{2}\sqrt{2}e^{-3t}\sin\sqrt{2}t \end{bmatrix} \xrightarrow{t \gg 1/3} \begin{bmatrix} 30 \\ 10 \\ 15 \end{bmatrix}$$

asymptotic solution

$\vec{x}' = A\vec{x}$ for closed 3 tank system

$$A = \begin{bmatrix} -k_1 & 0 & k_3 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

$$(k_1, k_2, k_3) = \left(\frac{r}{V_1}, \frac{r}{V_2}, \frac{r}{V_3}\right) = \left(\frac{60}{60}, \frac{60}{20}, \frac{60}{30}\right) = (1, 3, 2)$$