5.5.37 $y''' - 2y'' + y' = 1 + xe^x$ $\quad y(0)=0,\ y'(0)=0,\ y''(0)=1$

$r^3 - 2r^2 + r$
$= r(r^2 - 2r + 1)$
$= r(r-1)^2 = 0$
$r=0,\ m=1$
$r=1,\ m=2$
$y_h = c_1 e^{0x} + (c_2 + c_3 x)e^x$

$r=0,\ m=1$ $\quad r=1,\ m=2$ $\quad \Rightarrow \quad r(r-1)=0$
$D(D-1)(1+xe^x) = 0$

$D(D-1)^2 (D-1)^2 (D-0)\, y = D(D-1)(1+xe^x) = 0$

$r=1,\ m=4$
$r=0,\ m=2$

$y = (c_1 + c_4 x) + (c_2 + c_3 x + c_5 x^2 + c_6 x^3)e^x$
$= y_h + \underbrace{c_4 x + (c_3 x^2 + c_4 x^3)e^x}_{y_p = ax + (bx^2 + cx^3)e^x}$

$y_p = ax + (bx^2 + cx^3)e^x$
$y_p' = a + [(2bx + 3cx^2) + (bx^2 + cx^3)]e^x$
$= a + [2bx + (b+3c)x^2 + cx^3]e^x$
$y_p'' = [2bx + (b+3c)x^2 + cx^3] + [2b + 2(b+3c)x + 3cx^2]e^x$
$= [2b + (4b+6c)x + (b+6c)x^2 + cx^3]e^x$
$y_p''' = ([(4b+6c) + 2(b+6c)x + 3cx^2] + [2b + (4b+6c)x + (b+6c)x^2 + cx^3])e^x$
$= [(6b+6c) + (6b+18c)x + (b+9c)x^2 + cx^3]e^x$
$-2y_p'' = [-4b - (8b+12c)x - (2b+12c)x^2 - 2cx^3]e^x$
$+y_p' = a + [2bx + (b+3c)x^2 + cx^3]e^x$

$y_p''' + 2y_p'' + y_p' = a + \Big[\underbrace{(6-4)}_{2}b + 6c + \underbrace{[(6-8+2)}_{0}b + \underbrace{(18-12)}_{6}c]x + [\underbrace{(1-2+1)}_{0}b + \underbrace{(9-12+3)}_{0}c]x^2 + \underbrace{(1-2+1)}_{0}cx^3\Big]e^x$

$= a + [(2b+6c) + 6cx]e^x = 1 + xe^x$

$a=1,\ 2b+6c=0,\ 6c=1 \rightarrow c=1/6,\ b=-3c=-1/2 \rightarrow \boxed{y_p = x + \frac{1}{6}(x^3 - 3x^2)e^x}$

$y = c_1 + (c_2 + c_3 x)e^x + x + \frac{1}{6}(x^3 - 3x^2)e^x$
$y' = (c_3 + c_2 + c_3 x)e^x + 1 + \frac{1}{6}(3x^2 - 6x + x^3 - 3x^2)e^x \rightarrow \frac{1}{6}(x^3 - 6x)e^x$
$y'' = (c_3 + c_3 + c_2 + c_3 x)e^x + \frac{1}{6}(3x^2 - 6 + x^3 - 6x)$

$y(0) = c_1 + c_2 = 0$
$y'(0) = c_2 + c_3 + 1 = 0$
$y''(0) = c_2 + 2c_3 - 1 = 1$

$\begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{bmatrix}c_2\\c_3\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix},\ \begin{bmatrix}c_2\\c_3\end{bmatrix} = \frac{1}{2-1}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-2-2\\1+2\end{bmatrix} = \begin{bmatrix}-4\\3\end{bmatrix}$

$\therefore c_1 = -c_2 = 4$

$y = 4 + x + 3x + \frac{1}{6}(x^3 - 3x^2)e^x = \boxed{4 + x + \frac{1}{6}(x^3 - 3x^2 + 18x - 24)e^x}$ !! finally !!