

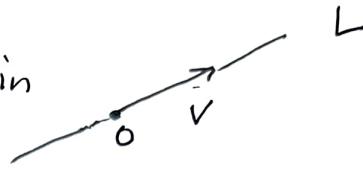
4.3

on  $\mathbb{R}^n$ :

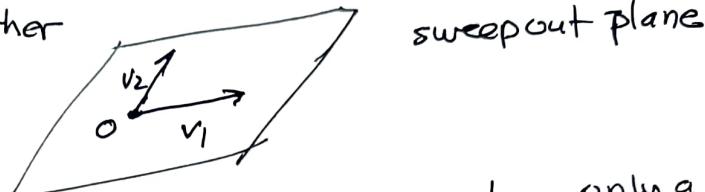
The set of all possible linear combinations  $c_1v_1 + \dots + c_pv_p$  for all  $c_1, \dots, c_p \in \mathbb{R}$  is called the span of this set:

$$\text{span } \{v_1, \dots, v_p\}.$$

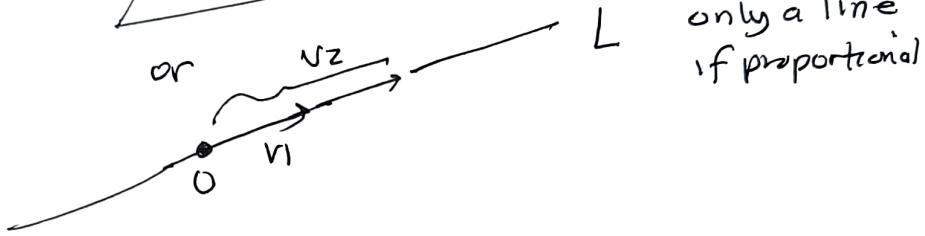
$p=1 \quad v \neq 0 : \quad \{cv\} = \text{line thru origin}$



$p=2 \quad v_1, v_2 \text{ nonzero either}$



or



only a line  
if proportional

etc.

These are exactly the subspaces of  $\mathbb{R}^n$  which are closed under vector addition & scalar multiplication or equivalently under linear combinations: any linear combination of vectors in a subspace is contained in that subspace.

The span of a set of vectors is automatically closed under linear combination.

Example  $a(c_1v_1+c_2v_2) + b(c_3v_1+c_4v_2)$

$\underbrace{\phantom{a(c_1v_1+c_2v_2) + b(c_3v_1+c_4v_2)}}_{\text{linear combs of } v_1, v_2}$

$$= \underbrace{ac_1v_1 + ac_2v_2}_{1} + \underbrace{bc_3v_1 + bc_4v_2}_{1}$$

$$= (ac_1 + bc_3)v_1 + (ac_2 + bc_4)v_2 \quad \text{again a linear comb of } v_1, v_2$$

We showed solns of  $AX=0$  closed under linear combination  
so they form subspaces.

The parametrized solns are the span of a set of vectors  
so form subspaces from that point of view too.

Exercise. Find the independent linear relationships among 4 vectors in  $\mathbb{R}^3$ :

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad u_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad u_4 = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

namely coefficients  $x_1, x_2, x_3, x_4$  such that

$$\rightarrow x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 = 0 \quad \text{namely:}$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 1 & 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \text{solve by row reduction technique}$$

$$\begin{bmatrix} 1 & 1 & 3 & 4 & 0 \\ 1 & 2 & 4 & 6 & 0 \\ 1 & 3 & 5 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 + 2x_3 + 2x_4 = 0 \rightarrow x_1 = -2t_1 - 2t_2 \\ x_2 + x_3 + 2x_4 = 0 \rightarrow x_2 = -t_1 - 2t_2 \\ 0 = 0 \end{array}$$

$\underbrace{x_1, x_2}_{\text{F}} \underbrace{x_3, x_4}_{\text{F}}$

$x_3 = t_1, x_4 = t_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t_1 - 2t_2 \\ -t_1 - 2t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

general solution  
for coefficients of columns

$$= t_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

backsub these  
values for  
 $x_1, x_2, x_3, x_4$

backsub these  
values for  
 $x_1, x_2, x_3, x_4$

Goal:

$$\begin{aligned} -2u_1 - 1u_2 + 1u_3 + 0u_4 &= 0 \\ -2u_1 - 2u_2 + 0u_3 + 1u_4 &= 0 \end{aligned}$$

these are the two independent  
'linear' relationships among  
 $u_1, u_2, u_3, u_4$

any other linear relationship is a  
linear combination of these two.

Explicitly:

$$-2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can always solve to express  
free columns in terms of  
the leading columns

$$u_3 = 2u_1 + u_2$$

$$u_4 = 2u_1 + 2u_2$$

these are obvious looking at  
the reduced matrix whose  
columns have the same  
relationships as the original  
columns (same coefficients  
 $x_1, x_2, x_3, x_4$ )

## Exercise (continued) 2

Note form of soln:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t_1 - 2t_2 \\ -t_1 - 2t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + t_2 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find lin comb equal to zero?

imply  
 $t_1 = 0$   
 $t_2 = 0$   
both coeffs forced to be zero  
so  
 $v_1, v_2$  lin. ind?

In general the vectors multiplying the free variable parameters are always linearly independent.

Algorithm leads to a lin. ind. set of vectors whose span is the soln space.

Suppose we want to express another vector  $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$  as a linear comb of  $\{u_1, u_2, u_3, u_4\}$ :

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 1 & 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \leftarrow \text{nonhomogeneous system}$$

$$= \begin{bmatrix} 7 \\ 10 \\ 13 \end{bmatrix}$$

but  $w$  must belong to  
Span  $\{u_1, u_2, u_3, u_4\}$   
to be written as a linear  
comb. of them!  
otherwise inconsistent system!  
not possible!

$$\text{Let } w = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 13 \end{bmatrix} = u_3 + u_4$$

$$\begin{bmatrix} 1 & 1 & 3 & 4 & 7 \\ 1 & 2 & 4 & 6 & 10 \\ 1 & 3 & 5 & 8 & 13 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \rightarrow \begin{aligned} x_1 &= 4 - 2t_1 - 2t_2 \\ x_2 &= 3 - t_1 - 2t_2 \\ x_3 &= t_1, x_4 = t_2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 2t_1 - 2t_2 \\ 3 - t_1 - 2t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -2t_1 - 2t_2 \\ -t_1 - 2t_2 \\ t_1 \\ t_2 \end{bmatrix}}_{X_h}$$

many ways to represent  $\langle 7, 10, 13 \rangle$   
in terms of 4 vectors!  
pick any values of  $t_1, t_2$ !

exercise continued (3)

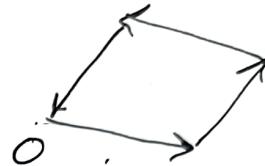
$$\begin{bmatrix} 7 \\ 10 \\ 13 \end{bmatrix} = (4-2t_1-2t_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (3-t_1-2t_2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t_4 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \quad (\text{separate})$$

$$= \underbrace{4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}}_{\text{unique linear comb of two leading cols}} + \underbrace{(2t_1-2t_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-t_1+2t_2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t_4 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}}$$

$$\left( = \begin{bmatrix} 4+t_3 \\ 4+6t_3 \\ 4+9t_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 13 \end{bmatrix} \vee \right)$$

but this = 0 so doesn't change preceding sum.

these sum to zero in many different ways



In general the leading columns are a linearly ind. subset which span the set of all columns and give a unique representation of any vector lying in that span.

On  $\mathbb{R}^n$  any linearly independent set of  $n$  vectors will span all of  $\mathbb{R}^n$  if their augmented matrix has nonzero determinant.

$$x_1 u_1 + x_2 u_2 + \dots + x_n u_n = w \quad \text{always has a unique soln}$$

$$\underbrace{\langle u_1 | u_2 | \dots | u_n \rangle}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \rightarrow x = \underbrace{A^{-1}w}_{\text{unique soln for coefficients}}$$

A  
if  $\det A \neq 0$ , invertible