

**Exercise.** Find the independent linear relationships among 4 vectors in  $\mathbb{R}^3$ :

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad u_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad u_4 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

namely coefficients  $x_1, x_2, x_3, x_4$  such that

$$x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 = 0 \quad \text{namely:}$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or  $\begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 1 & 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$  solve by row reduction technique

$$\begin{bmatrix} 1 & 1 & 3 & 4 & 0 \\ 1 & 2 & 4 & 6 & 0 \\ 1 & 3 & 5 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 + 2x_3 + 2x_4 = 0 \rightarrow x_1 = -2t_1 - 2t_2 \\ x_2 + x_3 + 2x_4 = 0 \rightarrow x_2 = -t_1 - 2t_2 \\ 0 = 0 \end{array}$$

$x_1, x_2, x_3, x_4$   
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$$x_3 = t_1, x_4 = t_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t_1 - 2t_2 \\ -t_1 - 2t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

general solution for coefficients of columns

$$= t_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

backsub these values for  $x_1, x_2, x_3, x_4$

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Goal:

$$\begin{array}{l} -2u_1 - 1u_2 + 1u_3 + 0u_4 = 0 \\ -2u_1 - 2u_2 + 0u_3 + 1u_4 = 0 \end{array}$$

these are the two independent linear relationships among  $u_1, u_2, u_3, u_4$

any other linear relationship is a linear combination of these two.

Explicitly:

$$-2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can always solve to express free columns in terms of the leading columns

$$u_3 = 2u_1 + u_2$$

$$u_4 = 2u_1 + 2u_2$$

these are obvious looking at the reduced matrix whose columns have the same relationships as the original columns (same coefficients  $x_1, x_2, x_3, x_4$ )