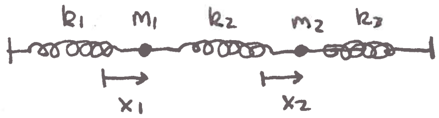
EBP3 7.4: 3,9 3 spring 2 mass system



$$m_1x_1'' = -(k_1+k_2)x_1 + k_2x_2 - c_1x_1' + F_1$$
$$m_2x_2'' = k_2x_1 - (k_2+k_3)x_2 - c_2x_2' + F_2$$

$m_1=1,\ m_2=2,\quad k_1=1,\ k_2=k_3=2,$

$c_1=c_2=0$ (no friction/damping)

$F_1=0,\ F_2=120\cos 3t$

PLUG IN:

$$\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}''}_{\vec{x}} = \underbrace{\begin{bmatrix}-3&2\\1&-2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix}0\\60\cos 3t\end{bmatrix}}_{\vec{f}}$$

$$\vec{x}'' = A\vec{x} + \vec{f}$$

initial conditions:
system starts out in equilibrium at rest
$x_1(0)=0=x_2(0),\quad x_1'(0)=0=x_2'(0).$

$$A=\begin{bmatrix}-3&2\\1&-2\end{bmatrix},\quad \begin{vmatrix}-3-\lambda&2\\1&-2-\lambda\end{vmatrix} = (\lambda+3)(\lambda+2)-2 = \lambda^2+5\lambda+4 = (\lambda+1)(\lambda+4)=0$$

$\lambda=-1,-4=\lambda_1,\lambda_2$

$\hookrightarrow\ B=\langle\vec{b}_1|\vec{b}_2\rangle = \begin{bmatrix}1&-2\\1&1\end{bmatrix}$

$B^{-1}=\frac{1}{3}\begin{bmatrix}1&2\\-1&1\end{bmatrix}$

$$\vec{x}_h = \underbrace{(c_1\cos t + c_2\sin t)\begin{bmatrix}1\\1\end{bmatrix}}_{\text{tandem mode}} + \underbrace{(c_3\cos 2t + c_4\sin 2t)\begin{bmatrix}-2\\1\end{bmatrix}}_{\text{accordian mode.}}$$

homogeneous soln

FIRST ORDER FORM

introduce "state vector" or I.C. vector

$$\vec{\mathcal{X}} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \equiv \begin{bmatrix}x_1\\x_2\\x_1'\\x_2'\end{bmatrix}\quad \text{so}\quad \vec{\mathcal{X}}' = \begin{bmatrix}x_1'\\x_2'\\x_3'\\x_4'\end{bmatrix} = \begin{bmatrix}x_3\\x_4\\x_1''\\x_2''\end{bmatrix} = \begin{bmatrix}x_3\\x_4\\-3x_1+2x_2\\x_1-2x_2+60\cos 3t\end{bmatrix} = \underbrace{\begin{bmatrix}0&0&1&0\\0&0&0&1\\-3&2&0&0\\1&-2&0&0\end{bmatrix}}_{\mathcal{A}}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \underbrace{\begin{bmatrix}0\\0\\0\\60\cos 3t\end{bmatrix}}_{\vec{\mathcal{F}}}$$

$x_1'=x_3$, $x_2'=x_4$ renaming

$$\boxed{\begin{aligned}\vec{\mathcal{X}}' &= \mathcal{A}\vec{\mathcal{X}} + \vec{\mathcal{F}}\\ \vec{\mathcal{X}}(0) &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\end{aligned}}$$

$$0=\det(\mathcal{A}-\lambda I) = \det\begin{bmatrix}-\lambda&0&1&0\\0&-\lambda&0&1\\-3&2&-\lambda&0\\1&-2&0&-\lambda\end{bmatrix} = \lambda^4+5\lambda^2+4 = (\lambda^2+1)(\lambda^2+4)$$

$\lambda^2=-1,-4$

$\lambda=i,-i,\ 2i,-2i$

$$\mathcal{B}=\langle\vec{b}_1|\vec{b}_2|\vec{b}_3|\vec{b}_4\rangle = \begin{bmatrix}-i&i&i&-i\\-i&i&-\frac{1}{2}i&\frac{1}{2}i\\1&1&-2&-2\\1&1&1&1\end{bmatrix}$$

$$\vec{\mathcal{X}}_h = c_1e^{it}\begin{bmatrix}-i\\-i\\1\\1\end{bmatrix} + \bar{c}_1e^{-it}\begin{bmatrix}i\\i\\1\\1\end{bmatrix} + c_3e^{2it}\begin{bmatrix}i\\-i/2\\-2\\1\end{bmatrix} + \bar{c}_3e^{-2it}\begin{bmatrix}-i\\i/2\\-2\\1\end{bmatrix} = a_1\begin{bmatrix}\sin t\\\sin t\\\cos t\\\cos t\end{bmatrix} + a_2\begin{bmatrix}-\cos t\\-\cos t\\\sin t\\\sin t\end{bmatrix} + a_3\begin{bmatrix}-\sin 2t\\\frac{1}{2}\sin 2t\\-2\cos 2t\\\cos 2t\end{bmatrix} + a_4\begin{bmatrix}\cos 2t\\-\frac{1}{2}\cos 2t\\-2\sin 2t\\\sin 2t\end{bmatrix} = \begin{bmatrix}\vec{x}_h\\\vec{x}_h'\end{bmatrix}$$

$$(\cos t + i\sin t)\begin{bmatrix}-i\\-i\\1\\1\end{bmatrix} = \begin{bmatrix}\sin t\\\sin t\\\cos t\\\cos t\end{bmatrix} + i\begin{bmatrix}-\cos t\\-\cos t\\\sin t\\\sin t\end{bmatrix},\quad (\cos 2t + i\sin 2t)\begin{bmatrix}i\\-i/2\\-2\\1\end{bmatrix} = \begin{bmatrix}-\sin 2t\\\frac{1}{2}\sin 2t\\-2\cos 2t\\\cos 2t\end{bmatrix} + i\begin{bmatrix}\cos 2t\\-\frac{1}{2}\cos 2t\\-2\sin 2t\\\sin 2t\end{bmatrix}$$

$$\vec{x}_h = (\underbrace{-a_2}_{d_1}\cos t + \underbrace{a_1}_{d_2}\sin t)\begin{bmatrix}1\\1\end{bmatrix} + (\underbrace{-\tfrac{a_4}{2}}_{d_3}\cos 2t + \underbrace{\tfrac{a_3}{2}}_{d_4}\sin 2t)\begin{bmatrix}-2\\1\end{bmatrix}$$

if had divided each eigenvector by second component to make it 1 → agreement with 2 vector direct solution eigenvectors

EXPERIMENT: what happens to homogeneous solution when $c_1=1,\ c_2=2$?
damping constants

E&P3 7.4(5): 3, 9 · 3 spring 2 mass system (reduction of order)

convenient rescaling of eigenvectors:
we are interested in the top two variables of $\vec{x} = \langle x_1, x_2, x_1', x_2' \rangle$
so we can choose eigenvectors which make those expressions simpler!

$$B = \begin{bmatrix} -i & i & i & -i \\ -i & i & -\frac{1}{2}i & \frac{1}{2}i \\ 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \end{bmatrix}}_{\text{rescale each column}} = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 1 \\ i & -i & -4i & 4i \\ i & -i & 2i & -2i \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_1 & \vec{b}_2 & \vec{b}_2 \\ i\vec{b}_1 & -i\vec{b}_1 & 2i\vec{b}_2 & -2i\vec{b}_2 \end{bmatrix}$$

$\vec{b}_1$ $\qquad$ $\vec{b}_2$

$$\vec{x}_h = \underbrace{c_1}_{\frac{c_1 - ic_2}{2}} e^{it}\begin{bmatrix} \vec{b}_1 \\ i\vec{b}_1 \end{bmatrix} + \underbrace{c_2}_{c_1+ic_2} e^{-it}\begin{bmatrix} \vec{b}_1 \\ -i\vec{b}_1 \end{bmatrix} + \underbrace{c_3}_{c_3 - ic_4} e^{2it}\begin{bmatrix} \vec{b}_2 \\ 2i\vec{b}_2 \end{bmatrix} + \underbrace{c_4}_{c_3+ic_4} e^{-2it}\begin{bmatrix} \vec{b}_2 \\ -2i\vec{b}_2 \end{bmatrix}$$

$$\left(\frac{c_1 - ic_2}{2}\right)(\cos t + i\sin t) = \frac{c_1\cos t + c_2 \sin t + i(\ldots)}{2}$$

the lower half is multiplied by $\pm i$ or $\pm 2i$ so that it forms the derivative of the upper half in the real part

$$= \begin{bmatrix} (c_1\cos t + c_2 \sin t)\vec{b}_1 \\ (-c_1\sin t + c_2\cos t)\vec{b}_1 \end{bmatrix} + \begin{bmatrix} (c_3\cos 2t + c_4 \sin 2t)\vec{b}_2 \\ (-2c_3\sin 2t + 2c_4 \cos 2t)\vec{b}_2 \end{bmatrix}$$

$$= 2\mathrm{Re}\left(c_1 e^{it}\begin{bmatrix} \vec{b}_1 \\ i\vec{b}_1 \end{bmatrix}\right) + 2\mathrm{Re}\left(c_3 e^{2it}\begin{bmatrix} \vec{b}_2 \\ 2i\vec{b}_2 \end{bmatrix}\right)$$

the "complexification" of the bottom half of the eigenvectors is necessary to allow complex multiplication to "take the derivative" of the complex exponential:

Thus $\vec{x} = (c_1\cos t + c_2 \sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3\cos 2t + c_4\sin 2t)\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\frac{d}{dt}\left(e^{i\omega t}\right) = i\omega e^{i\omega t}$$