3x3 matrix with repeated eigenvalues

$$A=\begin{bmatrix}9&4&0\\-6&-1&0\\6&4&3\end{bmatrix}\qquad A-\lambda I=\begin{bmatrix}9-\lambda&4&0\\-6&-1-\lambda&0\\6&4&3-\lambda\end{bmatrix}\xrightarrow{\det} -\lambda^3+11\lambda^2-39\lambda+45$$

$$= -(\lambda-5)(\lambda-3)^2=0 \quad \text{(factor)}$$

(solve)

$\lambda = 5,\ 3$

$m = 1,\ 2 \leftarrow$ multiplicity

$\underline{\lambda=5}$

$$A-5I=\begin{bmatrix}9-5&4&0\\-6&-1-5&0\\6&4&3-5\end{bmatrix}=\begin{bmatrix}4&4&0\\-6&-6&0\\6&4&-2\end{bmatrix}\to\begin{bmatrix}1&1&0\\3&2&-1\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&1&0\\0&-1&-1\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&1&0\\0&1&1\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\begin{array}{ccc}L&L&F\end{array}$$

$$\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1-x_3=0 \quad x_1=t$

$x_2+x_3=0 \quad x_2=-t$

$x_3=t$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\-t\\t\end{bmatrix}=t\underbrace{\begin{bmatrix}1\\-1\\1\end{bmatrix}}_{\vec{b}_1}$$

$\vec{b}_1 \to$ basis of 1d eigenspace

$\{\vec{b}_1,\vec{b}_2,\vec{b}_3\}$ basis of $\mathbb{R}^3$

$\underline{\lambda=3}$

$$A-3I=\begin{bmatrix}9-3&4&0\\-6&-1-3&0\\6&4&3-3\end{bmatrix}=\begin{bmatrix}6&4&0\\-6&-4&0\\6&4&0\end{bmatrix}\to\begin{bmatrix}1&2/3&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\{\vec{b}_2,\vec{b}_3\}$ basis of 2d eigenspace

$$\begin{array}{ccc}L&F&F\end{array}$$

$$\begin{bmatrix}1&\frac{2}{3}&0\\0&0&0\\0&0&0\end{bmatrix}$$

$x_1+\frac{2}{3}x_2=0 \quad x_1=-\frac{2}{3}t_2$

$x_2=t_1$

$x_3=t_2$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-\frac{2}{3}t_2\\t_1\\t_2\end{bmatrix}=t_1\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{b}_2}+t_2\begin{bmatrix}-2/3\\0\\1\end{bmatrix}$$

$\times 3 \ \vec{b}_3=\begin{bmatrix}-2\\0\\3\end{bmatrix}$ (integers!)

$$B=\langle\vec{b}_1|\vec{b}_2|\vec{b}_3\rangle=\begin{bmatrix}1&0&-2\\-1&1&0\\1&0&3\end{bmatrix}$$

compare Maple: $\begin{bmatrix}1&-2/3&0\\-1&1&0\\1&0&1\end{bmatrix}$

$\lambda=3$, maple does not care if fractions appear order unimportant.

$$B^{-1}=\frac{1}{5}\begin{bmatrix}3&0&2\\3&5&2\\-1&0&1\end{bmatrix}$$

change of basis:

$\vec{x}=B\vec{y},\ \vec{y}=B^{-1}\vec{x}$