Nonhomogeneous case: $\vec{x}\,' = A\vec{x} + \vec{F}$

$x_1' = x_2 + 2 \quad , \; x_1(0) = 1$

$x_2' = x_1 \quad , \; x_2(0) = 0$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2+2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1x_2 + 2 \\ 1x_1 + 0x_2 + 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\vec{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{F}}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Diagonalization $\quad \lambda = 1, -1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (\vec{b}_1 \; \vec{b}_2) \qquad B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad A_B = B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

change of basis

$$\vec{x} = y_1\vec{b}_1 + y_2\vec{b}_2 = B\vec{y} \; , \quad \vec{y} = B^{-1}\vec{x}$$

transform DE:

$$\vec{x}\,' = A\vec{x} + \vec{F} \longrightarrow \qquad (B\vec{y})' = A(B\vec{y}) + \vec{F} \rightarrow B^{-1}(B\vec{y}\,') = B^{-1}AB\vec{y} + B^{-1}\vec{F}$$

$$\vec{y}\,' = A_B\vec{y} + B^{-1}\vec{F}$$

new components of $\vec{F}$: $\quad B^{-1}\vec{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} y_1 + 1 \\ -y_2 - 1 \end{pmatrix}$$

solve decoupled DEs:

$y_1' = y_1 + 1 \; , \quad y_{1h} = c_1 e^t \qquad y_{1p} = c_3 \rightarrow \underbrace{(c_3)'}_{0} = c_3 + 1 \rightarrow c_3 = -1$

$y_2' = -y_2 + 1 \qquad y_{2h} = c_2 e^{-t} \qquad y_{2p} = c_4 \qquad \underbrace{(c_4)'}_{0} = -c_4 - 1 \rightarrow c_4 = -1$

$$\vec{y}_h = \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix} , \quad \vec{y}_p = \begin{bmatrix} -1 \\ -1 \end{bmatrix} , \quad \vec{y} = \vec{y}_h + \vec{y}_p = \begin{bmatrix} c_1 e^t - 1 \\ c_2 e^{-t} - 1 \end{bmatrix}$$

backsub:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^t - 1 \\ c_2 e^{-t} - 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \underbrace{c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{x}_p}$$

initial conditions:

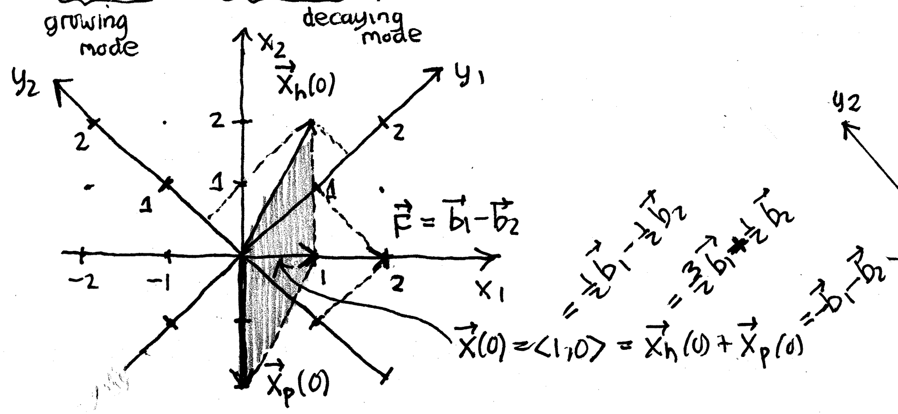
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{B}\begin{bmatrix} c_1 - 1 \\ c_2 - 1 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 - 1 \\ c_2 - 1 \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

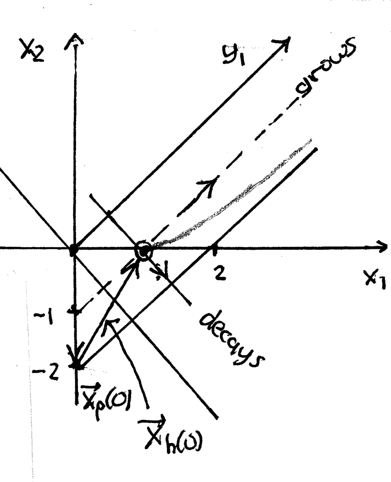
$c_1 - 1 = 1/2 \rightarrow c_1 = 3/2$

$c_2 - 1 = -1/2 \rightarrow c_2 = 1/2$

backsub:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\frac{3}{2}e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{growing mode}} + \underbrace{\frac{1}{2}e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{decaying mode}} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/2\,e^t - 1/2\,e^{-t} \\ 3/2\,e^t + 1/2\,e^{-t} - 2 \end{bmatrix} \qquad \vec{x}_h(0) = \begin{bmatrix} 3/2 - 1/2 \\ 3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$





2nd order Nonhomogeneous case : $\vec{x}'' = A\vec{x} + \vec{F}$

$x_1'' = x_2 + 2 \qquad x_1(0) = 1, \; x_1'(0) = 0$

$x_2'' = x_1 \qquad x_2(0) = 0, \; y_2'(0) = 1$

$$\Rightarrow \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} x_2+2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0x_1 + 1x_2 + 2 \\ 1x_1 + 0x_2 + 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{F}}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

diagonalization: $\lambda = 1, -1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\vec{b}_1 \; \vec{b}_2$

change of basis:

$$\vec{x} = y_1\vec{b}_1 + y_2\vec{b}_2 = B\vec{y}, \quad \vec{y} = B^{-1}\vec{x}$$

transform DE:

$$\vec{x}'' = A\vec{x} + \vec{F} \rightarrow (B\vec{y})'' = A(B\vec{y}) + \vec{F} \rightarrow B^{-1}(B\vec{y}'') = B^{-1}AB\vec{y} + B^{-1}\vec{F}$$

$$\vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}$$

new components of $\vec{F}$: $B^{-1}\vec{F} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} y_1 + 1 \\ -y_2 - 1 \end{bmatrix}$$

solve decouped 2nd order DEs:

$$\boxed{\begin{matrix} y_1'' = y_1 + 1 \\ y_2'' = -y_2 - 1 \end{matrix}} \rightarrow$$

$y_1'' - y_1 = 1 \quad y_{1h} = e^{rt} \rightarrow r^2 - 1 = 0, \; r = \pm 1, \; e^{rt} = e^{\pm t}, \quad y_{1h} = c_1e^t + c_2e^{-t}$

$y_2'' + y_2 = -1 \quad y_{2h} = e^{rt} \rightarrow r^2 + 1 = 0, \; r = \pm i, \; e^{rt} = e^{\pm it} = \cos t \pm i\sin t$

(unknowns on LHS ↖ nonhom driving terms)

$y_{2h} = c_3\cos t + c_4\sin t$

$y_{1p} = c_5 \qquad \underset{0}{(c_5)''} - c_5 = 1 \rightarrow c_5 = -1$

$y_{2p} = c_6 \qquad (c_6)'' + c_6 = -1 \rightarrow c_6 = -1$

$y_1 = y_{1h} + y_{1p} = c_1e^t + c_2e^{-t} - 1$

$y_2 = y_{2h} + y_{2p} = c_3\cos t + c_4\sin t - 1$

backsub:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^t + c_2e^{-t} - 1 \\ c_3\cos t + c_4\sin t - 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^t + c_2e^{-t} \\ c_3\cos t + c_4\sin t \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \underbrace{(c_1e^t + c_2e^{-t})\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3\cos t + c_4\sin t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\vec{x}_p}$$

initial conditions:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^t - c_2e^{-t} \\ -c_3\sin t + c_4\cos t \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 + c_2 - 1 \\ c_3 - 1 \end{bmatrix} \qquad \begin{bmatrix} c_1 + c_2 - 1 \\ c_3 - 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$c_1 + c_2 - 1 = 1/2$

$c_3 - 1 = -1/2 \rightarrow c_3 = 1 - 1/2 = 1/2$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 - c_2 \\ c_4 \end{bmatrix} \qquad \begin{bmatrix} c_1 - c_2 \\ c_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$c_1 - c_2 = 1/2$

$c_4 = 1/2 \rightarrow c_4 = 1/2$

$c_1 + c_2 = 1 + 1/2 = 3/2$

$c_1 - c_2 = 1/2$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/4 \\ 2/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{(e^t + \tfrac{1}{2}e^{-t})}_{\text{growth along } \vec{b}_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{(\tfrac{1}{2}\cos t + \tfrac{1}{2}\sin t)}_{\text{oscillation along } \vec{b}_2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} e^t + \frac{1}{2}e^{-t} - \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ e^t + \frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t - 2 \end{bmatrix}$$