extending the eigenvector decoupling technique

$\boxed{\text{1st order}}$ $\vec{x}' = A\vec{x}$ (homog) $\longrightarrow$ $\vec{x}' = A\vec{x} + \vec{F}$ (non-homog)

$\left.\begin{matrix}\vec{x} = B\vec{y} \\ \vec{y} = B^{-1}\vec{x}\end{matrix}\right\}$ $A_B = B^{-1}AB$ diagonal $\nearrow$ $\underbrace{B^{-1}\vec{x}'}_{\downarrow} = B^{-1}(A\overset{B\vec{y}}{\vec{x}} + \vec{F})$

$\vec{y}' = B^{-1}AB\vec{y} + B^{-1}\vec{F}$

decoupled but nonhom. eqns, easy to use method of undetermined coefficients for each scalar equation $\longleftarrow$ $\vec{y}' = A_B\vec{y} + B^{-1}\vec{F}$

$\longrightarrow \vec{y} = \vec{y}_h + \vec{y}_p$

$\vec{x} = B(\vec{y}_h + \vec{y}_p)$

$\boxed{\text{higher order (2}^{nd}\text{ enough)}}$

easy case: $\vec{x}'' = A\vec{x} \longrightarrow \vec{y}'' = A_B\vec{y} \longrightarrow y_i'' = \lambda_i y_i,\ y_i'' - \lambda_i y_i = 0$

$y_i = e^{r_i t} \longrightarrow r_i^2 - \lambda_i = 0 \rightarrow r_i = \pm\sqrt{\lambda_i}$

$y_i = a_i e^{\sqrt{\lambda_i}t} + b_i e^{-\sqrt{\lambda_i}t}$

$\vec{x} = B\vec{y}$

useful for undamped coupled oscillator systems and as above, easy to handle nonhomogeneous case $\vec{x}'' = A\vec{x} + \vec{F} \longrightarrow \vec{y}'' = A_B\vec{y} + B^{-1}\vec{F}$ with method of undetermined coefficients

less easy case: $\left[A_2\vec{x}'' + A_1\vec{x}' + A_0\vec{x} = \vec{F}\right]$ $\xrightarrow[\text{to standard form}]{\text{reduce}}$ $\vec{x}'' + A_2^{-1}A_1\vec{x}' + A_2^{-1}A_0\vec{x} = A_2^{-1}\vec{F}$

mult by A_2^{-1} $\det(A_2) \neq 0$ (otherwise mixed order)

relable

$\vec{x}'' + A_1\vec{x}' + A_0\vec{x} = \vec{F}$

can diagonalize one or the other but not both in general (different eigenvectors) so still coupled either way

solution: <u>reduction of order</u> express as 1st order deq for

initial data or "state" vector: $\mathcal{X} = \begin{bmatrix}\vec{x} \\ \vec{x}'\end{bmatrix} \longrightarrow \mathcal{X}' = \mathcal{A}\mathcal{X} + \vec{\mathcal{F}}$ solve as above

twice as many variables

double dimensions of matrix

(position & velocity at any time completely determine the "state" of a particle in motion)

> If A_2^{-1} does not exist get "mixed order" systems.
>
> Example: $x_1'' = x_2$ 2nd order in x_1: x_1, x_1'
>
> $x_2' = x_1$ 1st order in x_2: x_2

state vector: $\mathcal{X} = \begin{bmatrix}x_1 \\ x_2 \\ x_1'\end{bmatrix} \equiv \begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix}$

initial values required

$$\mathcal{X}' = \begin{bmatrix}x_1' \\ x_2' \\ x_3'\end{bmatrix} = \begin{bmatrix}x_3 \\ x_1 \\ x_1''\end{bmatrix} = \begin{bmatrix}x_3 \\ x_1 \\ x_2\end{bmatrix} \begin{matrix}\text{definition} \\ \text{D.E. 2} \\ \text{D.E. 1}\end{matrix} = \overbrace{\begin{bmatrix}0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}}^{\mathcal{A}}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} \leftrightarrow \mathcal{X}' = \mathcal{A}\mathcal{X},\ \mathcal{X}(0) = \begin{bmatrix}x_1(0) \\ x_2(0) \\ x_1'(0)\end{bmatrix}$$