

Damped Harmonic Oscillator Driven by Sinusoidal Driving Function

$$y'' + k_0 y' + \omega_0^2 y = B_0 \cos \omega t$$

$r = \pm i\omega$
 $r^2 + \omega^2 = 0$
 $D^2 + \omega^2$ annihilator

alternate approach to method undet. coeff.

$$(D^2 + \omega^2)(D^2 + k_0 D + \omega_0^2) y = 0$$

$\omega \neq \omega_0$
if $k_0 = 0$

$$\begin{cases} k_0 = \tau_0^{-1} > 0 \\ \omega_0 > 0 \\ \omega > 0 \\ B_0 > 0 \end{cases}$$

damping rate constant
natural oscillator freq.
driving freq.
driving amplitude

$$y = y_h + y_p$$

y_h = transient soln fixed by initial conditions
 $y_p = C_3 \cos \omega t + C_4 \sin \omega t$
= steady state solution

$$\omega_0^2 [y_p = C_3 \cos \omega t + C_4 \sin \omega t]$$

$$k_0 [Dy_p = -\omega C_3 \sin \omega t + \omega C_4 \cos \omega t]$$

$$1 [D^2 y_p = -\omega^2 C_3 \cos \omega t - \omega^2 C_4 \sin \omega t]$$

$$(D^2 + k_0 D + \omega_0^2) y_p = [(\omega_0^2 - \omega^2) C_3 + k_0 \omega C_4] \cos \omega t = B_0 \cos \omega t + 0 \sin \omega t$$

$$+ [-k_0 \omega C_3 + (\omega_0^2 - \omega^2) C_4] \sin \omega t$$

matrix soln for C_3, C_4

$$(\omega_0^2 - \omega^2) C_3 + k_0 \omega C_4 = B_0$$

$$-k_0 \omega C_3 + (\omega_0^2 - \omega^2) C_4 = 0$$

$$\begin{bmatrix} \omega_0^2 - \omega^2 & k_0 \omega \\ -k_0 \omega & \omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$\text{use: } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \omega_0^2 - \omega^2 & -k_0 \omega \\ k_0 \omega & \omega_0^2 - \omega^2 \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} B_0 \\ 0 \end{bmatrix} = \frac{B_0}{D} \begin{bmatrix} \omega_0^2 - \omega^2 \\ +k_0 \omega \end{bmatrix}$$

$$D = \det(D) = (\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2$$

$$y_p = \frac{B_0}{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2} \frac{[(\omega_0^2 - \omega^2) \cos \omega t + k_0 \omega \sin \omega t]}{\sqrt{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2}} \cos(\omega t - \delta)$$

$$= \frac{B_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + k_0^2 \omega^2}} \cos(\omega t - \delta)$$

A(ω) (divide numerator, denominator by ω_0^2)
to get dimensionless ratio ω/ω_0

■ response amplitude: $A(\omega) = \frac{B_0/\omega_0^2}{\sqrt{[1 - (\omega/\omega_0)^2]^2 + \frac{1}{(\omega_0 \tau_0)^2} (\omega/\omega_0)^2}}$

$$\frac{A(\omega)}{A(0)} = \frac{1}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + \frac{1}{(\omega_0 \tau_0)^2} (\omega/\omega_0)^2}}$$

approximate Quality factor:
 $Q = \omega_0 \tau_0$ determines shape of curve

compared to $\omega \rightarrow 0$
zero frequency limit of
constant driving function
 $B_0 \cos \omega t \rightarrow B_0$

$$\frac{A(\omega_0)}{A(0)} = \frac{1}{\sqrt{1/(Q\tau_0)^2}} = \omega_0 \tau_0 = Q \approx \text{peak value ratio for weakly damped system}$$

$Q \gg \frac{1}{2}$ (resonance)

■ response phase shift (lag):

$$\tan \delta = \frac{k_0 \omega}{\omega_0^2 - \omega^2} = \frac{1}{\omega_0 \tau_0} \cdot \frac{(\omega/\omega_0)}{1 - (\omega/\omega_0)^2} \rightarrow \delta \in [0, \pi] \text{ since } C_2 \geq 0$$

resonance: $\omega \rightarrow \omega_0$ means $\tan \delta \rightarrow \infty$ means $\delta \rightarrow \frac{\pi}{2}$, $\cos(\omega t - \frac{\pi}{2}) = \sin \omega t$

system lags by 90° but time derivative in phase with driving function since $y_p \sim \sin \omega t \rightarrow y_p' \sim \omega \cos \omega t$ in phase with $f \sim \cos \omega t$

so pushing in sync with direction system is moving

$\omega \gg \omega_0$, $\tan \delta \rightarrow 0$ while negative so $\delta \rightarrow \pi$, 180° out of phase: system can't keep up with driving function

Resonance in a damped harmonic oscillator driven by a sinusoidal forcing function

```
> deq := (D@@2)(y)(t)+k[0]*D(y)(t)+omega[0]^2*y(t)
= B[0]*cos(omega*t);
```

$$deq := (D^{(2)})(y)(t) + k_0 D(y)(t) + \omega_0^2 y(t) = B_0 \cos(\omega t)$$

```
> steady_state := B[0]/omega[0]^2*A_dimensionless*cos(omega*t-delta);
> steady_state :=  $\frac{B_0 A_{\text{dimensionless}} \cos(\omega t - \delta)}{\omega_0^2}$ 
```

Amplitude of response to sinusoidal driving function in units of the natural frequency as a function of the approximate "Quality factor" of the system:

```
> W:=omega/omega[0];
```

```
'Q:=omega[0]*tau[0];
```

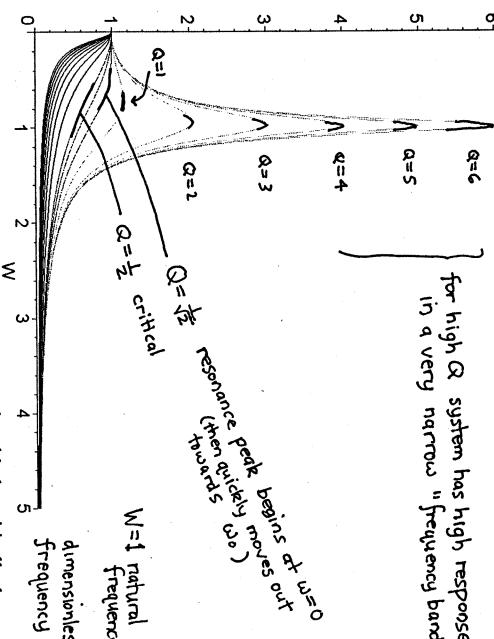
$$W = \frac{\omega}{\omega_0}$$

$$Q = \frac{\omega_0 \tau_0}{\omega_0}$$

```
> A_dimensionless:=unapply(1/sqrt((1-W^2)^2+W^2/Q^2), (W,Q));
A_dimensionless=A_dimensionless(W,Q);
```

$$A_{\text{dimensionless}} = \frac{1}{\sqrt{(1 - W^2)^2 + \frac{W^2}{Q^2}}}$$

for high Q system has high response
in a very narrow "frequency band"



The red zone is the slightly underdamped case with no resonance peak, with the critically damped case

as the lowest red curve. The overdamped cases have no resonance peak. Note how the resonance peak moves closer and closer to the natural frequency $W = 1$ as the Q factor increases (higher peaks). The green curves have $Q = 1..6$, while the blue curves have $Q = \frac{1}{3}..10$. Between $Q = \frac{1}{2}..1/\sqrt{2}$, the system is slightly underdamped but no resonance peak exists. [Set W-derivative to zero to locate local maximum.]

```
> phaseshift:=(W,Q)->arctan(W/Q, 1-W^2);
```

$\text{phaseshift} = (W, Q) \rightarrow \arctan\left(\frac{W}{Q}, 1 - W^2\right)$ i.e. $\arctan\left(\frac{\omega/\sqrt{Q}}{1 - \omega^2/Q}\right)$

```
> plot(Pi/Pi/2,phaseshift(W,1/40),phaseshift(W,1/2),seq(phaseshift(W,i),i=1..6),phaseshift(W,40),1,W=0..4,color=black);
```

$\text{phaseshift} = (W, Q) \rightarrow \arctan\left(\frac{W}{Q}, 1 - W^2\right)$ i.e. $\arctan\left(\frac{\omega/\sqrt{Q}}{1 - \omega^2/Q}\right)$



> plot commands

→ ω range
sharp frequency response in phaseshift
near resonance for high Q as in amplitude

where velocity variable and forcing function
are approximately in phase ("pumping")

$\omega_{\text{peak}} = \omega_0 \sqrt{1 - \frac{1}{2}Q^2}$ peak amplitude for steady state y
 $\omega_{\text{system}} = \omega_0 \sqrt{1 - \frac{1}{4}Q^2}$ transient "quasi-frequency"
 $\wedge \omega_0$ for undriven system
peak amplitude for steady state y'

all approximately
 ω_0 for large Q