Linear homogeneous 2^{nd} order DE with constant positive (nonnegative) coefficients

$Ay'' + By' + Cy = 0 \longrightarrow$ standard form : $\boxed{y'' + k_0 y' + \omega_0^2 y = 0}$ $\quad k_0, \omega_0 \geq 0$

- natural frequency ω_0 (when $k_0 = 0$) $\longrightarrow$ natural period $T_0 = 2\pi/\omega_0$
- exponential decay rate factor k_0 (when $\omega_0 = 0$) $\longrightarrow$ natural characteristic time $\tau_0 = 1/k_0$

case i) $k_0 = 0$ undamped case : $y'' + \omega_0^2 y = 0 \rightarrow r^2 + \omega_0^2 = 0 \rightarrow r = \pm i\omega_0$

$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t = A \cos(\omega_0 t - \delta)$ oscillations at natural frequency

case ii) $\omega_0 = 0$ pure damping case : $y'' + k_0 y' = 0 \rightarrow r^2 + k_0 r = r(r + k_0) = 0 \rightarrow r = 0, -k_0$

$y = c_1 + c_2 e^{-k_0 t} = c_1 + c_2 e^{-t/\tau_0}$ exponential decay to constant value

case iii) $k_0 > 0, \omega_0 > 0$ general case of "damped harmonic motion"

$r^2 + k_0 r + \omega_0^2 = 0 \rightarrow r = r_\pm = \dfrac{-k_0 \pm \sqrt{k_0^2 - 4\omega_0^2}}{2} = \dfrac{k_0}{2}\left(-1 \pm \sqrt{1 - 4\omega_0^2 \tau_0^2}\right) = -\dfrac{k_0}{2} \pm i\omega_0 \sqrt{1 - \dfrac{1}{4\omega_0^2 \tau_0^2}}$

Introduce the "quality factor": $Q = \omega_0 \tau_0 = \dfrac{\omega_0}{k_0}$ ratio of rate factors = dimensionless angle

THREE SUBCASES

- OVER DAMPED $\quad Q < \frac{1}{2}$ oscillation less important than damping

  2 real roots, both negative : $|r_+| < |r_-|$

  $\tau_+ > \tau_-$ (slower decay, faster decay)

  $y = c_1 e^{r_+ t} + c_2 e^{r_- t} = c_1 e^{-t/\tau_+} + c_2 e^{-t/\tau_-}$

  2 decaying exps!

- CRITICALLY DAMPED $\quad Q = \frac{1}{2}$ oscillation comparable to damping

  1 real negative root : $r = -k_0/2$

  $\tau = 2\tau_0$ twice pure damping case characteristic time

  $y = (c_1 + c_2 t) e^{-t/\tau}$

- UNDERDAMPED $\quad Q > \frac{1}{2}$ oscillation more important than damping

  2 complex roots : $r_\pm = -\underbrace{k_0/2}_{k} \pm i\omega$

  twice the characteristic time $\quad k = k_0/2, \ \tau = \frac{2}{k_0}$

  $\omega = \omega_0 \sqrt{1 - \dfrac{1}{4\omega_0^2 \tau_0^2}}$

  $= \omega_0 \underbrace{\sqrt{1 - \dfrac{1}{4Q^2}}}_{<1}$ shorter frequency compared to undamped case

  (longer period)

  $y = e^{-\frac{t}{2\tau_0}} (c_1 \cos \omega t + c_2 \sin \omega t)$

  $= \underbrace{A e^{-\frac{t}{2\tau_0}}}_{A(t)} \cos(\omega t - \delta)$

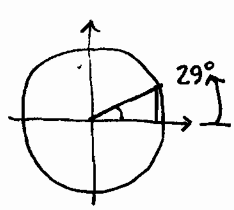
  $A(t)$ decaying amplitude

Critical damping condition

$\underbrace{Q = \omega_0 \tau_0}_{\text{phase angle during 1 decay time}} = \frac{1}{2}$ (radian) $\approx 29° \approx \frac{1}{12}$ cycle

frequency ↗ ↑ time

phase angle during 1 decay time



amount of phase angle of undamped oscillation that occurs during one decay time of the purely damped case

when it reaches 29° the coupled solution begins to have oscillating solutions with small frequency & long periods.

for high $Q \gg 1$, ω grows to almost the natural frequency: $\omega \approx \omega_0 (1 - \frac{1}{8Q^2})$

(using Taylor series approximation)

linear homogeneous 2^{nd} order DE ... damped harmonic oscillator (2)

underdamped case:

initial amplitude, frequency, phase shift

$$\left.\begin{array}{l} y'' + k_0 y' + \omega_0^2 y = 0 \\ Q = \omega_0 \tau_0 > \frac{1}{2} \end{array}\right\} \rightarrow y = A e^{-\frac{t}{\tau}} \cos(\omega t - \delta)$$

decay time

$$\tau = 2\tau_0$$
$$\omega = \omega_0 \sqrt{1 - 1/4Q^2}$$
$$T = 2\pi/\omega = \frac{2\pi}{\omega_0\sqrt{1-1/4Q^2}}$$

How can we turn these formulas into something we can see in the plot of these decaying sinusoidal functions?

Since $e^{-4.6} \approx 0.01$, we can only "see" full oscillations which occur roughly inside the window $t = 0..4\tau$, i.e., whose period T is less than about 4 decaytimes. The number of oscillations within this window is found by dividing the period into its length:

$$N = \begin{array}{l}\text{\# oscillations}\\ \text{during 4}\\ \text{decay times}\end{array} = \frac{4\tau}{T} = 4(2\tau_0)\left(\frac{\omega_0\sqrt{1-1/4Q^2}}{2\pi}\right) = \frac{4Q}{\pi}\sqrt{1-1/4Q^2} \approx \frac{4}{\pi}Q$$

$\sqrt{1-1/4Q^2} \to 0$ for $Q \gg 1$.

$Q = 1/2 \rightarrow N = 0$ (this is critically damped, no oscillations!)

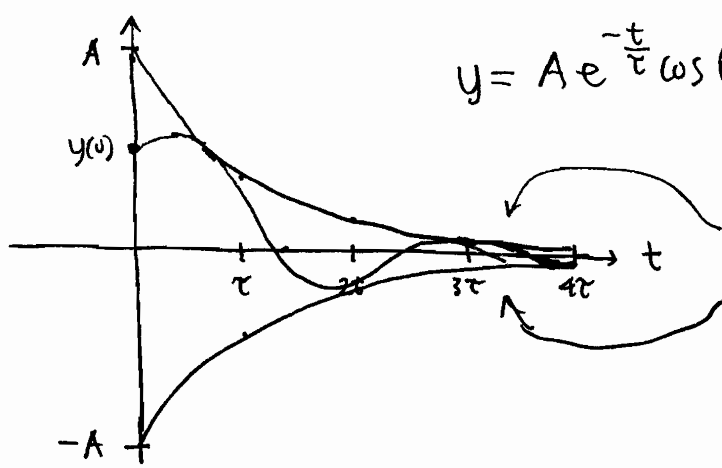
$Q = 1/\sqrt{2} \rightarrow N \approx 0.64$ (here $\omega = 1$, $T = 2\pi$)

$Q = 1 \rightarrow N \approx 1.1$

$Q = 2 \rightarrow N \approx 2.46$

$Q = 4 \rightarrow N \approx 5.1$

$Q = 8 \rightarrow N \approx 10.2$



$$y = A e^{-\frac{t}{\tau}} \cos(\omega t - \delta)$$

$$y = \pm A e^{-\frac{t}{\tau}}$$

exponential envelope of oscillation