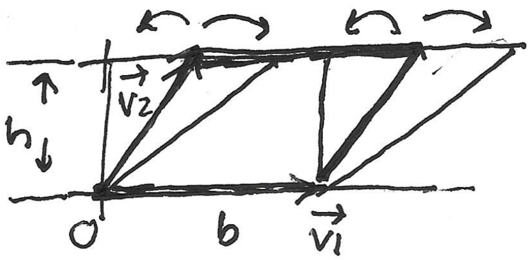
determinants evaluate areas of parallelograms (1)

and volumes of parallelopipeds in space ($\mathbb{R}^3$)
and "volumes" of n-parallelopipeds in $\mathbb{R}^n$

The 2-d example captures all the features of the higher dimensions.

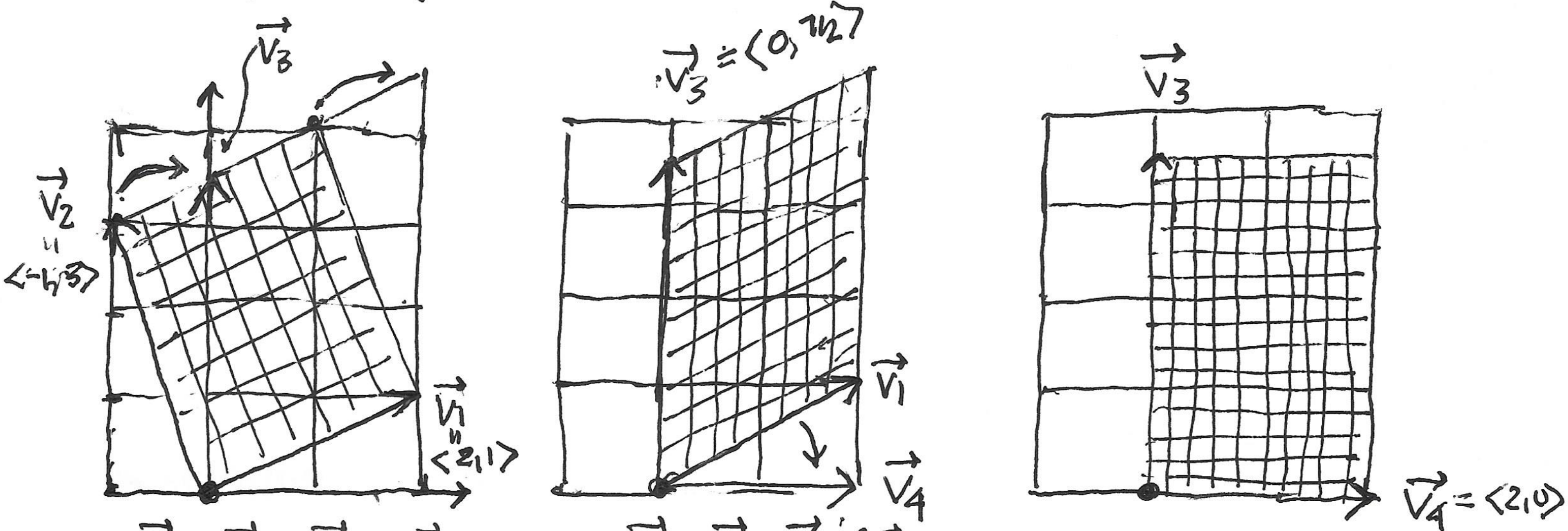


$A = bh$

sliding the tip of one side along that side does not change the height of the parallelogram relative to those parallel sides.

This corresponds to adding a multiple of $\vec{v}_1$ to $\vec{v}_2$ to move the tip of $\vec{v}_2$.
All these parallelograms have the same area as the one rektangle in this family

Let $V = \langle v_1 | v_2 \rangle$ augment the corresponding column matrices.
Then $V^T = \langle v_1^T, v_2^T \rangle$ makes these into rows.
The AddRow operation adds one vector multiple to another, exactly capturing this sliding activity, so does not change the area. But with 2 AddRow operations we can "diagonalize" the square matrix, which aligns the new sides with the coordinate axes, making a rectangle whose area is just the product of the final edge lengths.



$\vec{v}_2 \to \vec{v}_3 = \vec{v}_2 + \frac{1}{2}\vec{v}_1$ $\qquad$ $\vec{v}_1 \to \vec{v}_4 = \vec{v}_1 - \frac{2}{7}\vec{v}_3$

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \xrightarrow[R_2 \to R_2 + \frac{1}{2}R_1]{} \begin{bmatrix} v_1^T \\ v_3^T \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 7/2 \end{bmatrix} \xrightarrow[R_1 \to R_1 + \frac{2}{7}R_2]{} \begin{bmatrix} v_4^T \\ v_3^T \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 7/2 \end{bmatrix}$$

$+1 \quad +1/2$

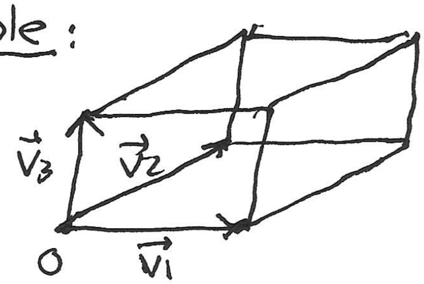
$\det = 6 + 1 = 7$ $\qquad$ $\det = 2(7/2) = 7$

obvious area!

$$\text{Area} = \det(\langle v_1 | v_2 \rangle) = \det(\langle v_1^T, v_2^T \rangle)$$

determinants evaluate volumes of parallelupipeds (2)

example:



$\vec{v_1} = \langle 1, 1, 1 \rangle$
$\vec{v_2} = \langle 2, 1, 3 \rangle$
$\vec{v_3} = \langle -1, 3, 1 \rangle$

AddRow ops dont change det value.

$$V^T = \langle V_1 | V_2 | V_3 \rangle^T = \begin{bmatrix} 1 & 1 & 1 \\ 2_{-2} & 1_{-2} & 3_{-2} \\ -1 & 3 & 1 \end{bmatrix} \begin{array}{c} \\ \\ \\ {\scriptstyle +1\ +1\ +1} \end{array} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{array}{c} {\scriptstyle +0\ +1\ +1} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 4 & 2 \end{bmatrix} \\ {\scriptstyle +0\ -4\ +4} \end{array}$$

$$\xrightarrow[R_3 \to R_3 + 4R_2]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_2 - \frac{1}{6}R_3]{R_1 \to R_1 - \frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

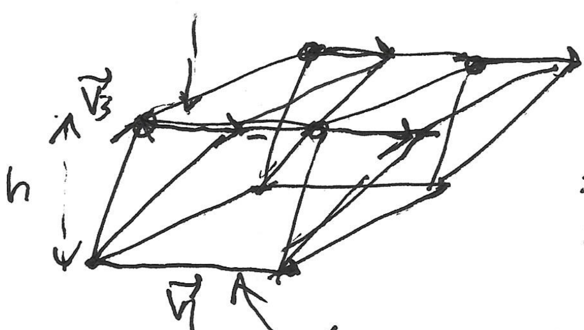
now a rectangular box extending respectively 1, 1, 6 units from the origin along the axes so Volume $= 6$

$\det(V^T) = -6$

sign irrelevant to volume

$k\vec{v_1}$

shift top face



adding a multiple of $\vec{v_1}$ to $\vec{v_3}$ only shifts the "upper" face parallel to itself so its height doesnt change, so volume doesnt change.

A = Area Parallelogram (base area)

Thus AddRow ops dont change volume.

Volume = (Area of base)(height) = Ah