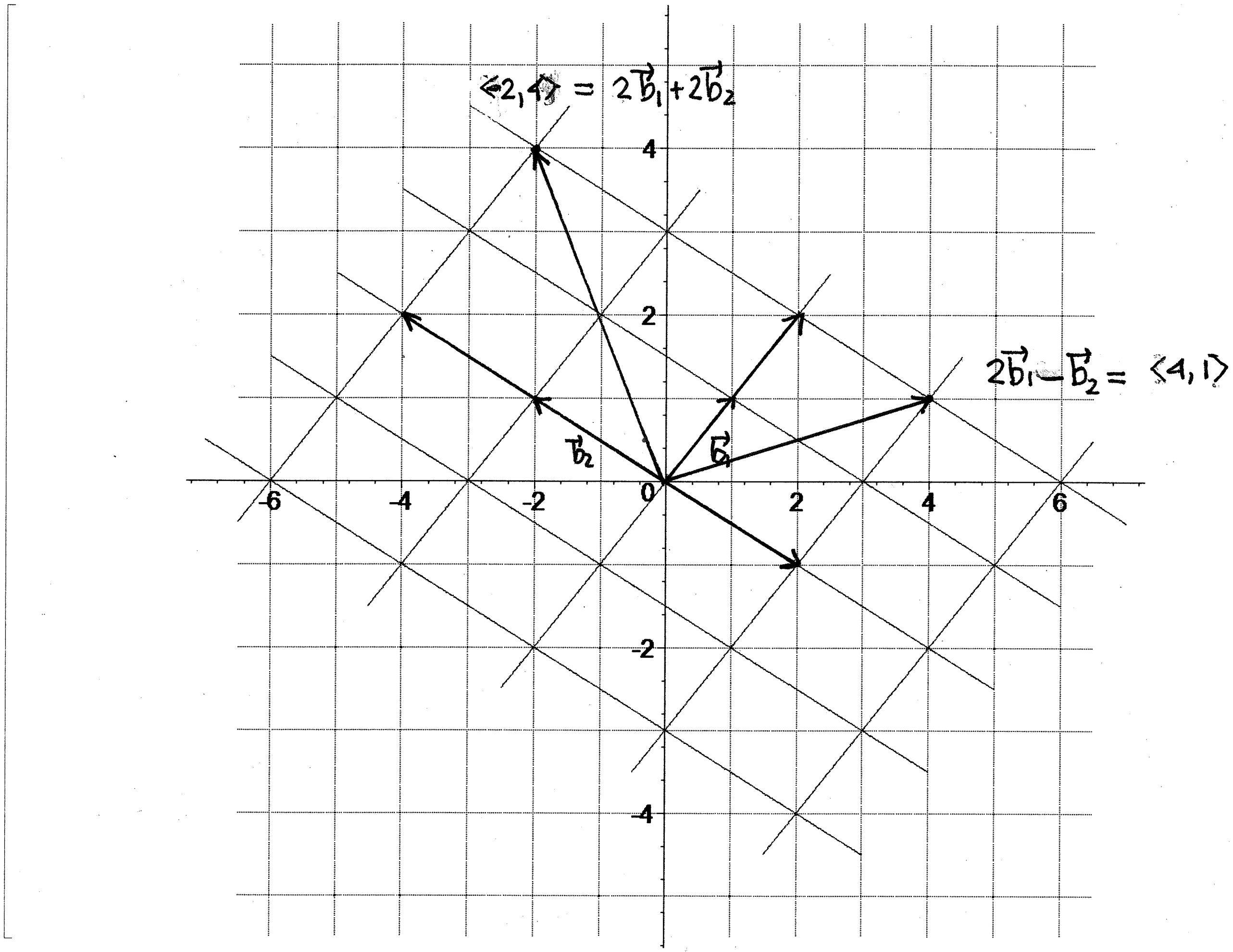
geometry of diagonalization



$\vec{b}_1 = \langle 1,1\rangle$, $\vec{b}_2 = \langle -2,1\rangle$ $\quad B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ $\quad B^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$\vec{x} = B\vec{y}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad x_1 = y_1 - 2y_2$, $x_2 = y_1 + y_2$

$\vec{y} = B^{-1}\vec{x}$: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad y_1 = \frac{1}{3}(x_1 + 2x_2)$, $y_2 = \frac{1}{3}(-x_1 + x_2)$

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$\lambda_1 = 5,\ \vec{b}_1 = \langle 1,1\rangle$
$\lambda_2 = -1,\ \vec{b}_2 = \langle -2,1\rangle$

(right click menu) $\longrightarrow$

$A_B = B^{-1} A B = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & +2 \\ 5 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 15 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ ✓

LinearAlgebra: Eigenvectors (A) $= \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ ordering of eigenvalues still random

$\lambda = -1 \quad \lambda = 5$ columns are corresponding eigenvectors in same order

1st order linear homogeneous DE system : real eigenvalues

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad \lambda = 5, -1$

$B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ (columns $\vec{b}_1$, $\vec{b}_2$) $\quad B^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad A_B = B^{-1}AB = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

$\vec{x}\,' = A\vec{x} \qquad \frac{dx_1}{dt} = x_1 + 4x_2 \qquad \xrightarrow{\vec{x} = B\vec{y}} \qquad \vec{y}\,' = A_B\vec{y} \qquad \frac{dy_1}{dt} = 5y_1 \rightarrow y_1 = c_1 e^{5t}$

$\qquad \frac{dx_2}{dt} = 2x_1 + 3x_2 \qquad \vec{y} = B^{-1}\vec{x} \qquad\qquad \frac{dy_2}{dt} = -y_2 \rightarrow y_2 = c_2 e^{-t}$

$\vec{x}(0) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ — $x_1(0) = -2,\ x_2(0) = 4 \qquad y(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftarrow y_1(0) = c_1,\ y_2(0) = c_2$

$\vec{x}(0) = B\vec{y}(0) \rightarrow \begin{bmatrix} -2 \\ 4 \end{bmatrix} = B\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

general soln:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{5t} \\ c_2 e^{-t} \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} - 2c_2 e^{-t} \\ c_1 e^{5t} + c_2 e^{-t} \end{bmatrix}$$

$$= c_1 \underbrace{e^{5t}}_{\text{growth along } \vec{b}_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \underbrace{e^{-t}}_{\text{decay along } \vec{b}_2} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

eigenvector entries set relative initial values for variables in each "mode" of exponential behavior

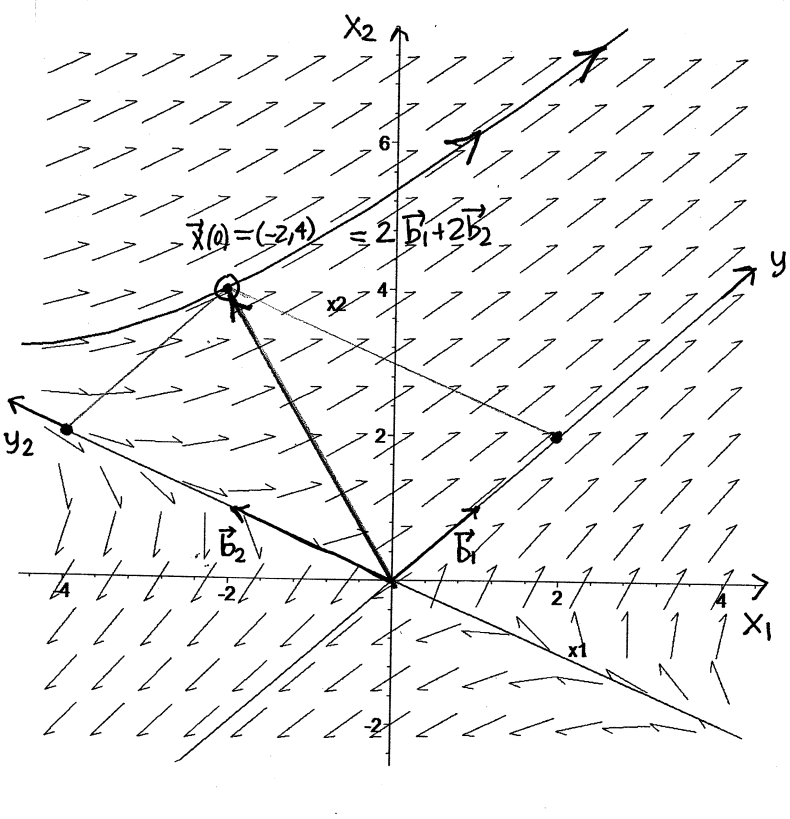
IVP soln:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2e^{5t} - 4e^{-t} \\ 2e^{5t} + 2e^{-t} \end{bmatrix}$$

$$= 2e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

initial data sets the "mixture" coefficients of the combination of modes

the eigenvalues are the exponential rate coefficients for each mode



the decaying mode ($\lambda = -1$) decays to 1% of its original value in the time interval $t = 0..4.6$ during which the growing mode ($\lambda = 5$) grows by a factor $e^{5(4.6)} \approx 10^{10}$ (very big)