

1st order linear homogeneous DE system: complex eigenvalues (1)

$A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}$      $\lambda = -1 \pm 2i$ ,  $-1 - 2i = k \pm iw \rightarrow e^{\lambda t} = e^{(k \pm iw)t} = e^{kt} e^{\pm iwt} = e^{-t} e^{\pm 2it}$   
 $\rightarrow e^{-t} \cos 2t, e^{-t} \sin 2t$   
 $B = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ 1 & 1 \end{bmatrix}$      $B^{-1} = \begin{bmatrix} i & \frac{1}{2} + \frac{i}{2} \\ -i & \frac{1}{2} - \frac{i}{2} \end{bmatrix}$      $A_B = B^{-1}AB = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$

$\vec{X}' = A\vec{X}$

$\frac{dx_1}{dt} = x_1 + 2x_2$      $\vec{X} = B\vec{y}$      $\vec{y}' = A_B\vec{y}$      $\frac{dy_1}{dt} = (-1+2i)y_1 \rightarrow y_1 = c_1 e^{(-1+2i)t}$   
 $\frac{dx_2}{dt} = -4x_1 - 3x_2$      $\vec{y} = B^{-1}\vec{X}$      $\frac{dy_2}{dt} = (-1-2i)y_2 \rightarrow y_2 = c_2 e^{(-1-2i)t}$

→ real, imaginary parts are exp. modulated sinusoids

initial conditions:  $\vec{X}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow x_1(0)=1, x_2(0)=2$      $\vec{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \leftarrow y_1(0)=c_1, y_2(0)=c_2$

direct complex soln:  $\vec{X}(0) = B\vec{y}(0) \rightarrow \vec{y}(0) = B^{-1}\vec{X}(0)$      $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1-2i \end{bmatrix}$  ← Note  $c_2 = \bar{c}_1$

general soln:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} e^{2ti} \\ c_2 e^{-t} e^{-2ti} \end{bmatrix} = c_1 e^{-t} e^{2ti} \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix} + c_2 e^{-t} e^{-2ti} \begin{bmatrix} -\frac{1}{2}(1-i) \\ 1 \end{bmatrix}$

↑ real → twice real part of first term → complex soln since  $z + \bar{z} = 2\text{Re}(z)$     its complex conjugate if  $c_2 = \bar{c}_1$

BUT FORMULA NOT EXPLICITLY REAL!    REMEDY: take real, imaginary parts of this complex vector soln

$e^{-t} e^{2ti} \begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -\frac{1}{2}(1+i)(\cos 2t + i \sin 2t) \\ \cos 2t + i \sin 2t \end{bmatrix}$  → complex multiplication!  
 $= e^{-t} \begin{bmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ \cos 2t \end{bmatrix} + i e^{-t} \begin{bmatrix} -\frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t \\ \sin 2t \end{bmatrix}$

these are 2 independent real vector solns of the DE, use as an explicitly real basis of soln space

general soln REAL:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a e^{-t} \begin{bmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ \cos 2t \end{bmatrix} + b e^{-t} \begin{bmatrix} -\frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t \\ \sin 2t \end{bmatrix}$     NO MORE i!

initial conditions:

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$     no need for matrix soln  
 $= \begin{bmatrix} -\frac{1}{2}a - \frac{1}{2}b \\ a \end{bmatrix} \rightarrow \begin{matrix} \text{solve} & a=2 \\ & b=-2(1)-a=-4 \end{matrix}$     easy

IVP soln:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 e^{-t} \begin{bmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t \\ \cos 2t \end{bmatrix} - 4 e^{-t} \begin{bmatrix} -\frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t \\ \sin 2t \end{bmatrix} \stackrel{\text{combine}}{=} e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2 \cos 2t - 4 \sin 2t \end{bmatrix}$

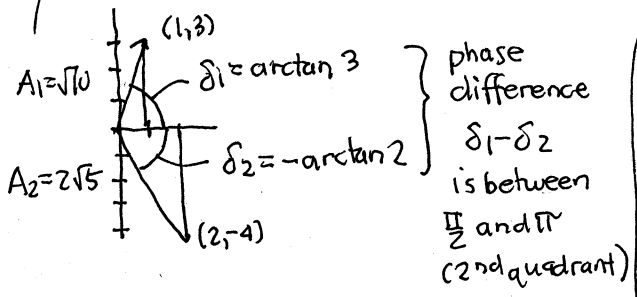
we could have gotten this directly by backsubstituting the complex coefficients into the complex soln:  
 $c_2 = 1 - 2i = \frac{1}{2}(a + ib) = \frac{1}{2}(2 - 4i) \checkmark$   
 but we would have to evaluate products of 3 complex numbers

# 1st order linear homogeneous DE system: complex eigenvalues - INTERPRETATION (2)

Both  $x_1$  and  $x_2$  are exponentially modulated sinusoidal functions  $\sim e^{kt} (\cos \omega t, \sin \omega t)$   
 whose rate parameters  $k$  &  $\omega$  are determined by the complex eigenvalues  $\lambda = k \pm i\omega$   
 What are the amplitudes and phase shifts of the sinusoidal factors?  $-\delta_1 = -\delta_2 - (\delta_1 - \delta_2)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2 \cos 2t - 4 \sin 2t \end{bmatrix} = e^{-t} \begin{bmatrix} \sqrt{10} \cos(2t - \delta_1) \\ 2\sqrt{5} \cos(2t - \delta_2) \end{bmatrix} = \sqrt{10} e^{-t} \begin{bmatrix} \frac{1}{\sqrt{2}} \cos(2t - \delta_2 - 3\pi/4) \\ \frac{1}{2} \cos(2t - \delta_2) \end{bmatrix}$$

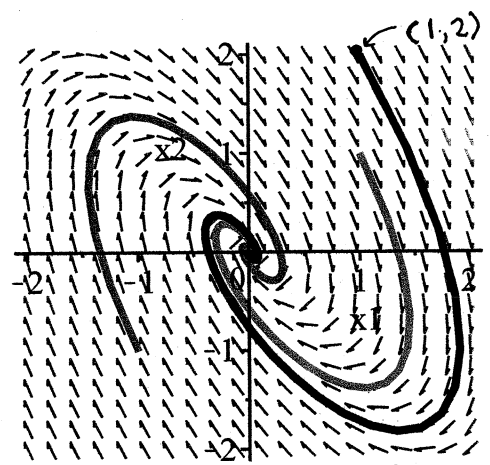
amplitude ratio:  $\frac{A_1}{A_2} = \frac{\sqrt{10}}{2\sqrt{5}} = \frac{1}{\sqrt{2}}$



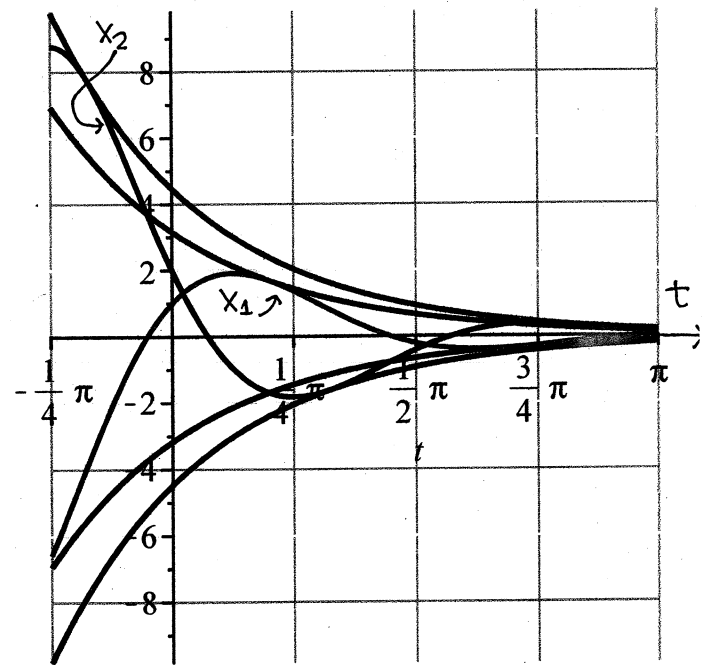
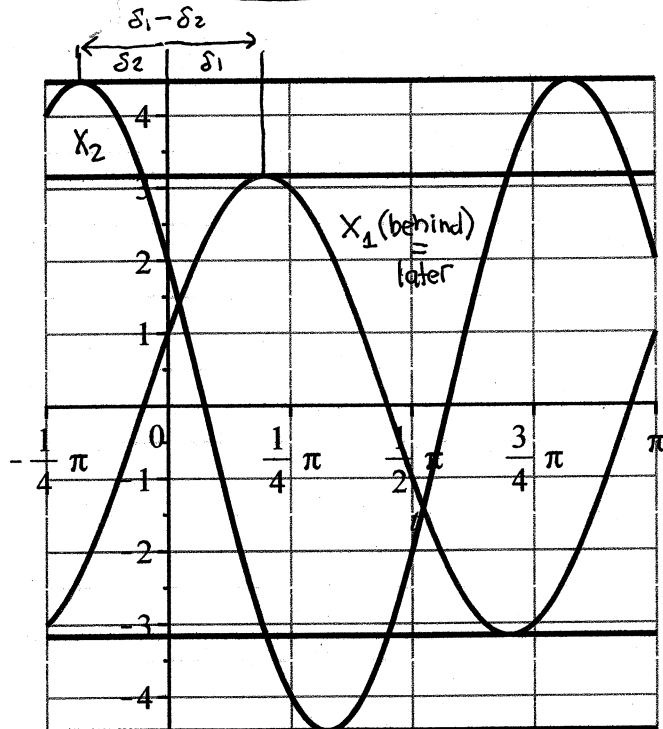
conclusion:  $x_1$  is  $\frac{3\pi}{4}$  phase later than  $x_2$  (to the right, behind in time) and  $\frac{1}{\sqrt{2}}$  smaller in amplitude

$\delta_1 - \delta_2 = \arctan 3 - (-\arctan 2)$   
 $\tan(\delta_1 - \delta_2) = \tan(\arctan 3 + \arctan 2)$   
 $\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$   
 $= \frac{3 + 2}{1 - 3(2)} = \frac{5}{-5} = -1$   
 2nd quad soln:  $\delta_1 - \delta_2 = 3\pi/4$

ouch!



$x_1$  and  $x_2$  spiral inward here  $\rightarrow$



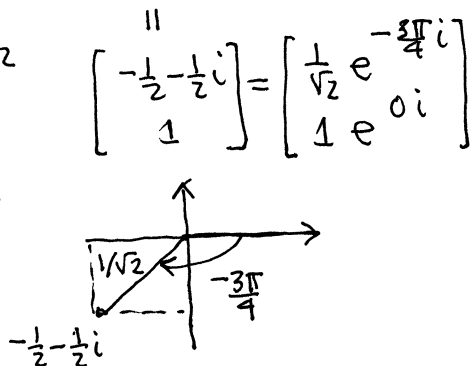
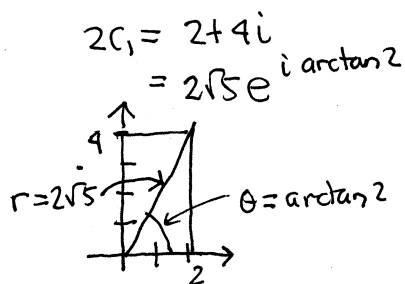
The sinusoidal factors in  $x_1$  and  $x_2$  are locked together by the DE, with a fixed amplitude ratio and fixed overall phase, i.e., you can only stretch or compress the left graph vertically or shift it left or right, which is controlled by the initial conditions.

# 1st order linear homogeneous DE system: complex eigenvalues - INTERPRETATION (3)

The complex eigenvalues determine the exponential rate factor and sinusoidal frequency. The complex eigenvectors determine the relative amplitude and relative phase of the variables  $x_1$  and  $x_2$ .

Expressing the complex factors (i) the complex soln in polar form makes their product easy to evaluate.

$$\vec{x} = \text{Re} \left( 2c_1 e^{-t} e^{2ti} \vec{b}_1 \right)$$



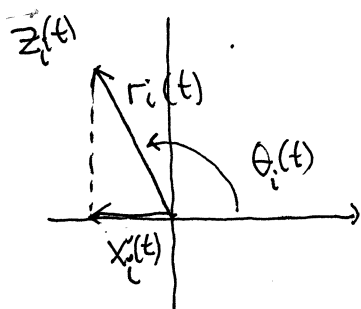
polar form details

$$\vec{x} = \text{Re} \left( \underbrace{2\sqrt{5} e^{i \arctan 2}}_{\text{initial conditions set overall scale \& phase}} e^{-t} e^{2ti} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}i} \\ 1 \end{bmatrix}}_{\text{eigenvector sets relative amplitudes \& phases}} \right) = \text{Re}(\vec{Z})$$

$$= e^{-t} \text{Re} \left( \begin{bmatrix} \sqrt{10} e^{i(2t + \arctan 2 - 3\pi/4)} \\ 2\sqrt{5} e^{i(2t + \arctan 2)} \end{bmatrix} \right)$$

$$= e^{-t} \begin{bmatrix} \sqrt{10} \cos(2t + \arctan 2 - 3\pi/4) \\ 2\sqrt{5} \cos(2t + \arctan 2) \end{bmatrix}$$

(exponential products are easiest than trigonometric combination formulas)



the complex solutions  $Z_i(t)$  show amplitude and phase and the real part in the complex plane visually.

used in electrical engineering, physics, etc.

(beyond scope of our course)