1st order linear homogeneous DE system : complex eigenvalues (1)

$$A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \qquad \lambda = -1+2i\ ,\ -1-2i = k \pm i\omega \rightarrow e^{\lambda t} = e^{(k\pm i\omega)t} = e^{kt}e^{\pm i\omega t} = e^{-t}e^{\pm 2it}$$

$$\hookrightarrow e^{-t}\cos 2t,\ e^{-t}\sin 2t$$

$$B = \begin{bmatrix} \underbrace{\begin{matrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{matrix}}_{\vec{b}_1} & \underbrace{\begin{matrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{matrix}}_{\vec{b}_2} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} i & \frac{1}{2}+\frac{i}{2} \\ -i & \frac{1}{2}-\frac{i}{2} \end{bmatrix} \qquad A_B = B^{-1}AB = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$$

$$\vec{x}\,' = A\vec{x}$$

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + 2x_2 \\ \frac{dx_2}{dt} &= -4x_1 - 3x_2 \end{aligned} \quad \xrightarrow[\vec{y} = B^{-1}\vec{x}]{\vec{x} = B\vec{y}} \quad \vec{y}\,' = A_B\vec{y} \qquad \begin{aligned} \frac{dy_1}{dt} &= (-1+2i)y_1 \rightarrow y_1 = c_1 e^{(-1+2i)t} \\ \frac{dy_2}{dt} &= (-1-2i)y_2 \rightarrow y_2 = c_2 e^{(-1-2i)t} \end{aligned}$$

→ real, imaginary parts are exp. modulated sinusoidals

initial conditions: $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow x_1(0) = 1,\ x_2(0) = 2$ $\qquad \vec{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \leftarrow y_1(0) = c_1,\ y_2(0) = c_2$

(direct complex soln: $\vec{x}(0) = B\vec{y}(0) \rightarrow \vec{y}(0) = B^{-1}\vec{x}(0)$ $\quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 1-2i \end{bmatrix} \leftarrow$ Note $c_2 = \overline{c_1}$)

$\boxed{\text{general soln:}}$ $\vec{x} = B\vec{y}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{-t}e^{2ti} \\ c_2 e^{-t}e^{-2ti} \end{bmatrix} = \underbrace{c_1 e^{-t}e^{2ti}\begin{bmatrix} -\frac{1}{2}(1+i) \\ 1 \end{bmatrix}}_{\text{complex soln}} + \underbrace{c_2 e^{-t}e^{-2ti}\begin{bmatrix} -\frac{1}{2}(1-i) \\ 1 \end{bmatrix}}_{\text{its complex conjugate if } c_2 = \overline{c_1}}$$

real → twice real part of first term → complex soln

since $z + \bar{z} = 2\,\mathrm{Re}(z)$

BUT FORMULA NOT EXPLICITLY REAL! REMEDY: take real, imaginary parts of this complex vector soln

$$e^{-t}e^{2ti}\begin{bmatrix} -\frac{1}{2}(1-i) \\ 1 \end{bmatrix} = e^{-t}\begin{bmatrix} -\frac{1}{2}(1-i)(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t \end{bmatrix}$$

complex multiplication!

$$= \underbrace{e^{-t}\begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \\ \cos 2t \end{bmatrix}}_{} + i\,\underbrace{e^{-t}\begin{bmatrix} -\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \\ \sin 2t \end{bmatrix}}_{}$$

these are 2 independent real vector solns of the DE, use as an explicitly real basis of soln space

$\boxed{\text{general soln REAL:}}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\,e^{-t}\begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \\ \cos 2t \end{bmatrix} + b\,e^{-t}\begin{bmatrix} -\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \\ \sin 2t \end{bmatrix}$$

NO MORE i!

initial conditions:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

no need for matrix soln

$$= \begin{bmatrix} -\frac{1}{2}a - \frac{1}{2}b \\ a \end{bmatrix} \xrightarrow{\text{solve}} \begin{aligned} a &= 2 \\ b &= -2(1) - a = -4 \end{aligned}$$

easy

$\boxed{\text{IVP soln!}}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2e^{-t}\begin{bmatrix} -\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \\ \cos 2t \end{bmatrix} - 4e^{-t}\begin{bmatrix} -\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \\ \sin 2t \end{bmatrix} \overset{\text{combine}}{=} e^{-t}\begin{bmatrix} \cos 2t + 3\sin 2t \\ 2\cos 2t - 4\sin 2t \end{bmatrix}$$

(we could have gotten this directly by backsubstituting the complex coefficients into the complex soln:
$c_2 = 1 - 2i = \frac{1}{2}(a + ib) = \frac{1}{2}(2 - 4i)$ ✓
but we would have to evaluate products of 3 complex numbers)

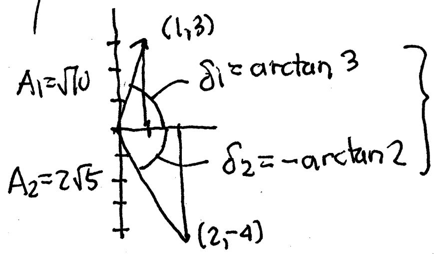
1st order linear homogeneous DE system: complex eigenvalues — INTERPRETATION (2)

Both x_1 and x_2 are exponentially modulated sinusoidal functions $\sim e^{kt}(\cos\omega t, \sin\omega t)$ whose rate parameters k & ω are determined by the complex eigenvalues $\lambda = k \pm i\omega$

What are the amplitudes and phase shifts of the sinusoidal factors?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-t}\begin{bmatrix} \cos 2t + 3\sin 2t \\ 2\cos 2t - 4\sin 2t \end{bmatrix} = e^{-t}\begin{bmatrix} \sqrt{10}\cos(2t-\delta_1) \\ 2\sqrt{5}\cos(2t-\delta_2) \end{bmatrix} = \sqrt{10}\, e^{-t}\begin{bmatrix} \frac{1}{\sqrt{2}}\cos(2t \overbrace{-\delta_2 - 3\pi/4}^{-\delta_1 = -\delta_2 - (\delta_1-\delta_2)}) \\ 1\cos(2t-\delta_2) \end{bmatrix}$$

amplitude ratio: $\frac{A_1}{A_2} = \frac{\sqrt{10}}{2\sqrt{5}} = \frac{1}{\sqrt{2}}$



phase difference $\delta_1 - \delta_2$ is between $\frac{\pi}{2}$ and π (2nd quadrant)

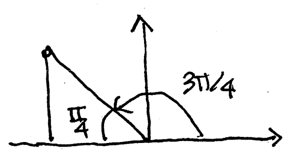
conclusion: x_1 is $\frac{3\pi}{4}$ phase later than x_2 (to the right, behind in time) and $\frac{1}{\sqrt{2}}$ smaller in amplitude

$\delta_1 - \delta_2 = \arctan 3 - (-\arctan 2)$

$\tan(\delta_1-\delta_2) = \tan(\arctan 3 + \arctan 2)$

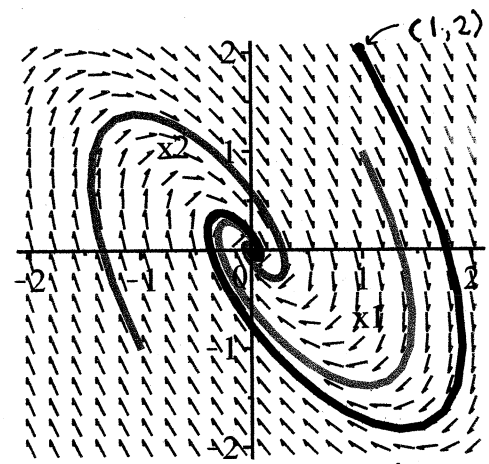
$= \frac{3+2}{1-3(2)} = \frac{5}{-5} = -1$

$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1-\tan\theta_1\tan\theta_2}$

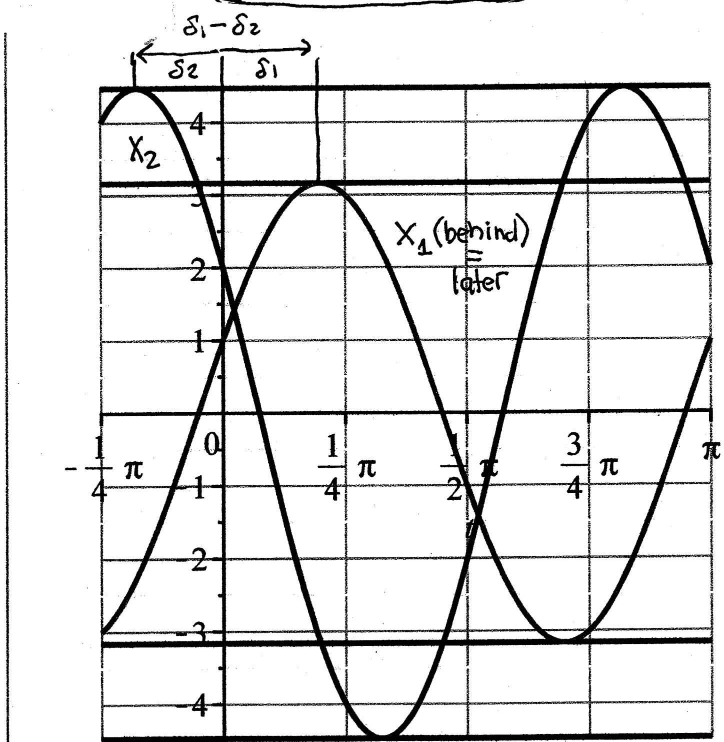


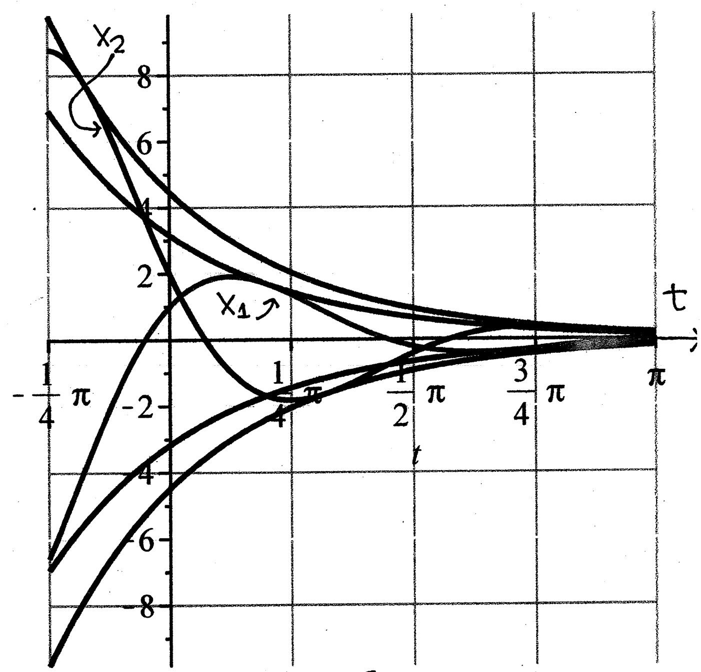
ouch!

2nd quad soln: $\delta_1 - \delta_2 = 3\pi/4$



x_1 and x_2 spiral inward here





The sinusoidal factors in x_1 and x_2 are locked together by the D.E. with a fixed amplitude ratio and fixed overall phase, i.e., you can only stretch or compress the left graph vertically or shift it ~~left~~ left or right, which is controlled by the initial conditions.

1st order linear homogeneous DE system: complex eigenvalues — INTERPRETATION (3)

The complex eigenvalues determine the exponential rate factor and sinusoidal frequency.
The complex eigenvectors determine the relative amplitude and relative phase of the variables x_1 and x_2.

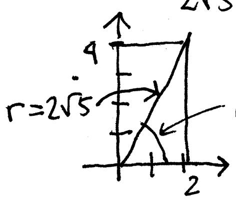
Expressing the complex factors in the complex soln in polar form makes their product easy to evaluate.

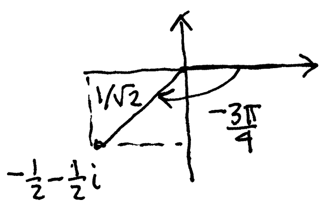
$$\vec{x} = \mathrm{Re}\left(\underbrace{2c_1}\, e^{-t} e^{2ti} \underbrace{\vec{b}_1}\right)$$

$$2c_1 = 2+4i = 2\sqrt{5}\, e^{i \arctan 2}$$

$$\begin{bmatrix} -\frac{1}{2}-\frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4} i} \\ 1\, e^{0i} \end{bmatrix}$$

polar form details



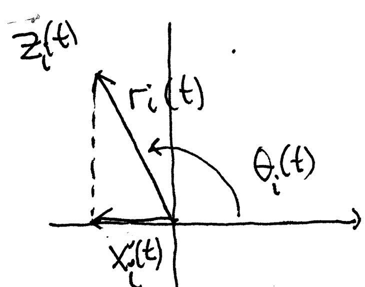


$$\vec{x} = \mathrm{Re}\left(\underbrace{2\sqrt{5}\, e^{i \arctan 2}}_{\text{initial conditions set overall scale \& phase}} e^{-t} e^{2ti} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4} i} \\ 1 \end{bmatrix}}_{\text{eigenvector sets relative amplitudes \& phases}}\right) = \mathrm{Re}(\vec{z}\,)$$

$$= e^{-t}\, \mathrm{Re}\left(\begin{bmatrix} \sqrt{10}\, e^{i(2t + \arctan 2 - 3\pi/4)} \\ 2\sqrt{5}\, e^{i(2t + \arctan 2)} \end{bmatrix}\right)$$

$$= e^{-t} \begin{bmatrix} \sqrt{10} \cos(2t + \arctan 2 - 3\pi/4) \\ 2\sqrt{5} \cos(2t + \arctan 2) \end{bmatrix}$$

(exponential products are easiest than trigonometric combination formulas)



the complex solutions $z_i(t)$ show amplitude and phase and the real part in the complex plane visually.
used in electrical engineering, physics, etc.

(beyond scope of our course)

From a complex to a real basis of solutions

Suppose $z(t)$ and $\bar{z}(t)$ are complex solutions of a real linear homogeneous differential equation. Namely $z(t) = x(t) + i\,y(t)$, $\bar{z}(t) = x(t) - i\,y(t)$ which contain only 2 real functions.

change of variables:

$$z = x + iy$$
$$\bar{z} = x - iy$$

sum: $z + \bar{z} = 2x \to x = \frac{1}{2}(z+\bar{z}) = \mathrm{Re}(z) = \mathrm{Re}(\bar{z})$

diff: $z - \bar{z} = 2iy \to y = \frac{1}{2i}(z-\bar{z}) = \mathrm{Im}(z) = -\mathrm{Im}(\bar{z})$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{bmatrix}}_{B} \begin{bmatrix} z \\ \bar{z} \end{bmatrix}$$ linear change of coordinates

$\to \det B = -\frac{1}{2i} \neq 0$ so x and y are linearly independent combinations of z and $\bar{z}$

Thus any linear combination of $z, \bar{z}$ can be expressed as a linear combination of x, y and vice versa.

CONCLUSION:

If a soln technique produces solutions $z(t)$, $\bar{z}(t)$ so that by linearity $C_1 z(t) + C_2 \bar{z}(t)$ is also a solution, then instead one can use $c_1 x(t) + c_2 y(t)$. If c_1, c_2 are real, we get an explicitly real solution formula.

EX. $e^{(a+ib)t}, e^{(a-ib)t} \longrightarrow e^{at}\cos bt, e^{at}\sin bt$ (Re and Im parts of positive imaginary part exponential)

EX $e^{(a+ib)t}\vec{b}_1, e^{(a-ib)t}\vec{\bar{b}}_1 \to \mathrm{Re}(e^{(a+ib)t}\vec{b}_1), \mathrm{Im}(e^{(a+ib)t}\vec{b}_1)$

suppose $\vec{b}_1 = \vec{u} + i\vec{v}$

$$e^{(a+ib)t} b_1 = e^{at}(\cos bt + i \sin bt)(\vec{u} + i\vec{v})$$
$$= \underbrace{e^{at}(\cos bt\,\vec{u} - \sin bt\,\vec{v})}_{\substack{\text{Re part} \\ \vec{B}_1(t)}} + i\,\underbrace{e^{at}(\sin bt\,\vec{u} + \cos bt\,\vec{v})}_{\substack{\text{Im part} \\ \vec{B}_2(t)}}$$

real soln: $c_1 \vec{B}_1(t) + c_2 \vec{B}_2(t)$

The only difference is that now we are dealing with complex vector solutions of a real linear homogeneous vector differential equation.