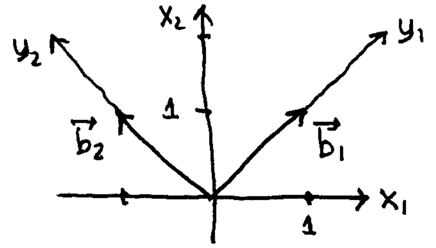
1st and (simple) 2nd order linear DE systems compared

$$A=\begin{bmatrix}0&4\\4&0\end{bmatrix}\quad 0=\det(A-\lambda I)=\det\begin{bmatrix}-\lambda&4\\4&-\lambda\end{bmatrix}=\lambda^2-16=(\lambda-4)(\lambda+4)\ \to\ \lambda=4,-4$$

$$\lambda=4:\ A-\lambda I=\begin{bmatrix}-4&4\\4&-4\end{bmatrix}\to\overset{\begin{matrix}L&F\end{matrix}}{\begin{bmatrix}1&-1\\0&0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\quad \begin{matrix}x_1-x_2=0\\0=0\end{matrix}\quad \begin{matrix}x_1=x_2=t\\x_2=t\end{matrix}\quad \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}t\\t\end{bmatrix}=t\begin{bmatrix}1\\1\end{bmatrix}\quad \underline{b}_1=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\lambda=-4:\ A-\lambda I=\begin{bmatrix}4&4\\4&4\end{bmatrix}\to\overset{\begin{matrix}L&F\end{matrix}}{\begin{bmatrix}1&1\\0&0\end{bmatrix}}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\quad \begin{matrix}x_1+x_2=0\\0=0\end{matrix}\quad \begin{matrix}x_1=-x_2=-t\\x_2=t\end{matrix}\quad \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-t\\t\end{bmatrix}=t\begin{bmatrix}-1\\1\end{bmatrix}\quad \underline{b}_2=\begin{bmatrix}-1\\1\end{bmatrix}$$

$$B=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\quad B^{-1}=\frac{1}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\quad A_B=B^{-1}AB=\begin{bmatrix}4&0\\0&-4\end{bmatrix}$$



$$\underline{x}=B\underline{y}=y_1\underline{b}_1+y_2\underline{b}_2=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}y_1-y_2\\y_1+y_2\end{bmatrix}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\underline{y}=B^{-1}\underline{x}=\frac{1}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\frac12(x_1+x_2)\\\frac12(-x_1+x_2)\end{bmatrix}=\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

■ a) $\underline{x}'=\underline{A}\,\underline{x}$

$$\begin{bmatrix}x_1'\\x_2'\end{bmatrix}=\begin{bmatrix}0&4\\4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\qquad \begin{matrix}x_1'=4x_2\\x_2'=4x_1\end{matrix}\quad \text{scalar form}$$

$$\downarrow$$

$$B^{-1}(B\underline{y})'=B^{-1}A(B\underline{y})\qquad y_i=e^{rt}$$

$$\downarrow$$

$$\underline{y}'=A_B\underline{y}\qquad \begin{bmatrix}y_1'\\y_2'\end{bmatrix}=\begin{bmatrix}4&0\\0&-4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}\quad \begin{matrix}y_1'=4y_1\\y_2'=-4y_2\end{matrix}\ \to\ \begin{matrix}r-4=0\\r+4=0\end{matrix}\quad \begin{matrix}r=4\\r=-4\end{matrix}\quad \begin{matrix}y_1=c_1e^{4t}\\y_2=c_2e^{-4t}\end{matrix}\quad \begin{matrix}\text{growth along } \vec{b_1}\\\text{decay along } \vec{b_2}\end{matrix}$$

$$\underline{x}=\underbrace{c_1e^{4t}\begin{bmatrix}1\\1\end{bmatrix}}_{\text{1 real mode}}+\underbrace{c_2e^{-4t}\begin{bmatrix}-1\\1\end{bmatrix}}_{\text{1 real mode}}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}c_1e^{4t}\\c_2e^{-4t}\end{bmatrix}=\begin{bmatrix}c_1e^{4t}-c_2e^{-4t}\\c_1e^{4t}+c_2e^{-4t}\end{bmatrix}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\qquad \text{general soln}$$

initial values $\underline{x}(0)=\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}$ $\qquad \begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}\qquad \begin{bmatrix}c_1\\c_2\end{bmatrix}=\frac12\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

$\leftrightarrow\ y_1(0)=0,\ y_2(0)=2$

$$\underline{x}=\begin{bmatrix}e^{4t}-e^{-4t}\\e^{4t}+e^{-4t}\end{bmatrix}=\begin{bmatrix}2\sinh 4t\\2\cosh 4t\end{bmatrix}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\qquad \text{IVP soln}$$

■ b) $\underline{x}''=\underline{A}\,\underline{x}$

$$\begin{bmatrix}x_1''\\x_2''\end{bmatrix}=\begin{bmatrix}0&4\\4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\qquad \begin{matrix}x_1''=4x_2\\x_2''=4x_1\end{matrix}\quad \text{scalar form}$$

$$\downarrow$$

$$\underline{B}^{-1}(B\underline{y})''=B^{-1}A(B\underline{y})\qquad y_i=e^{rt}$$

$$\downarrow$$

$$\underline{y}''=A_B\underline{y}\qquad \begin{bmatrix}y_1''\\y_2''\end{bmatrix}=\begin{bmatrix}4&0\\0&-4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}\quad \begin{matrix}y_1''=4y_1\\y_2''=-4y_2\end{matrix}\quad \begin{matrix}y_1''-4y_1=0\\y_2''+4y_2=0\end{matrix}\ \to\ \begin{matrix}r^2-4=0\\r^2+4=0\end{matrix}\quad \begin{matrix}r=2,-2\\r=2i,-2i\end{matrix}\quad \begin{matrix}(=\pm\sqrt{\lambda_1})\\(=\pm\sqrt{\lambda_2})\end{matrix}$$

$$y_1=c_1e^{2t}+c_2e^{-2t}\ \text{(growth/decay)},\quad y_2=c_3\cos 2t+c_4\sin 2t\quad \text{(oscillation)}$$

$$\underline{x}=\underbrace{(c_1e^{2t}+c_2e^{-2t})\begin{bmatrix}1\\1\end{bmatrix}}_{\text{2 real modes}}+\underbrace{(c_3\cos 2t+c_4\sin 2t)\begin{bmatrix}-1\\1\end{bmatrix}}_{\text{one complex mode}}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}c_1e^{2t}+c_2e^{-2t}\\c_3\cos 2t+c_4\sin 2t\end{bmatrix}=\begin{bmatrix}c_1e^{2t}+c_2e^{-2t}-c_3\cos 2t-c_4\sin 2t\\c_1e^{2t}+c_2e^{-2t}+c_3\cos 2t+c_4\sin 2t\end{bmatrix}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

general soln

$$\begin{bmatrix}x_1'\\x_2'\end{bmatrix}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}2c_1e^{2t}-2c_2e^{-2t}\\-2c_3\sin 2t+2c_4\cos 2t\end{bmatrix}$$

initial values $\underline{x}(0)=\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}$, $\quad \underline{x}'(0)=\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}x_1'(0)\\x_2'(0)\end{bmatrix}$,

$\leftrightarrow\ \begin{matrix}y_1(0)=0, & y_1'(0)=-1\\y_2(0)=2 & y_2'(0)=1\end{matrix}$

$$\underline{x}(0)=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}c_1+c_2\\c_3\end{bmatrix},\quad \begin{bmatrix}c_1+c_2\\c_3\end{bmatrix}=\frac12\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\ \to c_3=1$$

$$\underline{x}'(0)=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}2c_1-2c_2\\2c_4\end{bmatrix},\quad \begin{bmatrix}2c_1-2c_2\\2c_4\end{bmatrix}=\frac12\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\ \to c_4=1/2$$

$$\begin{matrix}c_1+c_2=1\\2c_1-2c_2=0\end{matrix}\quad \begin{bmatrix}1&1&1\\2&-2&0\end{bmatrix}\to\begin{bmatrix}1&0&1/2\\0&1&1/2\end{bmatrix}\quad \begin{matrix}c_1=1/2\\c_2=1/2\end{matrix}$$

easier to just solve 4×4 system by row reduction!

$$\underline{x}=\underbrace{\frac12(e^{2t}+e^{-2t})}_{\cosh 2t}\begin{bmatrix}1\\1\end{bmatrix}+(\cos 2t+\tfrac12\sin 2t)\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}\cosh 2t-(\cos 2t+\frac12\sin 2t)\\\cosh 2t+(\cos 2t+\frac12\sin 2t)\end{bmatrix}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\qquad \text{IVP soln}$$

1st and (simple) 2nd order linear DE systems compared (2)

c) 2nd order reduced to 1st order $\quad A = \begin{bmatrix}0&4\\4&0\end{bmatrix}$

$\underline{x}'' = \underline{A}\,\underline{x}$ or $(\underline{x}')' = \underline{A}\,\underline{x}$ introduce state vector $\mathcal{X} = \begin{bmatrix}\underline{x}\\ \underline{x}'\end{bmatrix} \to \mathcal{X}' = \begin{bmatrix}\underline{x}'\\ (\underline{x}')'\end{bmatrix} = \begin{bmatrix}\underline{x}'\\ \underline{A}\,\underline{x}\end{bmatrix} = \begin{bmatrix}\underline{0} & \underline{1}\\ \underline{A} & \underline{0}\end{bmatrix}\begin{bmatrix}\underline{x}\\ \underline{x}'\end{bmatrix}$

$= \mathcal{A}\mathcal{X}$

or in scalar form:

$$x_1'' = 4x_2 \qquad x_2'' = 4x_1$$

DEF: $x_3 = x_1'$, $x_4 = x_2'$ $\longrightarrow$ $x_1' = x_3$, $x_2' = x_4$

DE: $x_3' = x_1'' = 4x_2$, $x_4' = x_2'' = 4x_1$ $\longrightarrow$ $x_3' = 4x_2$, $x_4' = 4x_1$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}' = \left[\begin{array}{cc|cc}0&0&1&0\\0&0&0&1\\ \hline 0&4&0&0\\4&0&0&0\end{array}\right]\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$\mathcal{X}' = \mathcal{A}\,\mathcal{X}$$

Note

$\mathcal{X} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_1'\\x_2'\end{bmatrix} = \begin{bmatrix}\underline{x}\\ \underline{x}'\end{bmatrix}$ is the state vector used to specify initial conditions: $\mathcal{X}(0) = \begin{bmatrix}x_1(0)\\x_2(0)\\x_1'(0)\\x_2'(0)\end{bmatrix} = \begin{bmatrix}0\\2\\-1\\1\end{bmatrix}$ $\longleftarrow$ IVP.

eigenvalues / eigenvectors:

$$0 = \det(\mathcal{A} - \lambda I) = \det\begin{bmatrix}-\lambda&0&1&0\\0&-\lambda&0&1\\0&4&-\lambda&0\\4&0&0&-\lambda\end{bmatrix} = \lambda^4 - 16 = (\lambda^2)^2 - 16 = (\lambda^2-4)(\lambda^2+4)$$
$$= (\lambda-2)(\lambda+2)(\lambda-2i)(\lambda+2i)$$

$\lambda \to \lambda^2$ in characteristic equation for A, eigenvalues of $\mathcal{A}$ are square roots of those of A, doubling in number

$A:\ \lambda = 4, -4$

$\mathcal{A}:\ \lambda = 2, -2, 2i, -2i$

$$\mathcal{A}_{\mathcal{B}} = \mathcal{B}^{-1}\mathcal{A}\mathcal{B} = \begin{bmatrix}2&0&0&0\\0&-2&0&0\\0&0&2i&0\\0&0&0&-2i\end{bmatrix}$$

$\mathcal{B} = \begin{bmatrix}1&1&i/2&-i/2\\1&1&-i/2&i/2\\2&-2&-1&-1\\2&-2&1&1\end{bmatrix}$ eigenvectors from MAPLE

$\mathcal{X} = \mathcal{B}\mathcal{Y}$ $\qquad \mathcal{B}^{-1}(\mathcal{B}\mathcal{Y})' = \mathcal{B}^{-1}(\mathcal{A}\mathcal{Y})$

$\mathcal{Y} = \mathcal{B}\mathcal{X}$ $\qquad \mathcal{Y}' = \mathcal{A}_{\mathcal{B}}\mathcal{Y}$ ← new DEs

$y_1' = 2y_1 \quad y_1 = c_1 e^{2t}$
$y_2' = -2y_2 \quad y_2 = c_2 e^{-2t}$
$y_3' = 2iy_3 \quad y_3 = c_3 e^{2it}$
$y_4' = -2iy_4 \quad y_4 = c_4 e^{-2it}$

$$\mathcal{X} = \underbrace{c_1 e^{2t}\begin{bmatrix}1\\1\\2\\2\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}1\\1\\-2\\-2\end{bmatrix}}_{\text{2 real modes}} + \underbrace{c_3 e^{2it}\begin{bmatrix}i/2\\-i/2\\-1\\1\end{bmatrix} + c_4 e^{-2it}\begin{bmatrix}-i/2\\i/2\\-1\\1\end{bmatrix}}_{\text{one complex mode}}$$

general solution

one complex mode ↓ explicitly real

$$e^{2it}\begin{bmatrix}i/2\\-i/2\\-1\\1\end{bmatrix} = \begin{bmatrix}(\cos 2t + i\sin 2t)(i/2)\\(\cos 2t + i\sin 2t)(-i/2)\\(\cos 2t + i\sin 2t)(-1)\\(\cos 2t + i\sin 2t)(1)\end{bmatrix} = \begin{bmatrix}-\frac12\sin 2t\\ \frac12\sin 2t\\ -\cos 2t\\ \cos 2t\end{bmatrix} + i\begin{bmatrix}\frac12\cos 2t\\ -\frac12\cos 2t\\ -\sin 2t\\ \sin 2t\end{bmatrix} \longrightarrow a\begin{bmatrix}-\frac12\sin 2t\\ \frac12\sin 2t\\ -\cos 2t\\ \cos 2t\end{bmatrix} + b\begin{bmatrix}\frac12\cos 2t\\ -\frac12\cos 2t\\ -\sin 2t\\ \sin 2t\end{bmatrix}$$

$$\therefore\ \underline{x} = (c_1 e^{2t} + c_2 e^{-2t})\begin{bmatrix}1\\1\end{bmatrix} + \left(\left(-\frac{b}{2}\right)\cos 2t + \frac{a}{2}\sin 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}$$

same as before $\qquad$ "c_3" ← previous "c_4" page

Bottom two entries of $\mathcal{X}$ give $\underline{x}'$.

initial conditions:

$$\mathcal{X}(0) = \begin{bmatrix}1&1&0&1/2\\1&1&0&-1/2\\2&-2&-1&0\\2&-2&1&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\a\\b\end{bmatrix} = \begin{bmatrix}0\\2\\-1\\1\end{bmatrix} \xrightarrow[\text{rref backsub}]{\text{augment}} \begin{bmatrix}c_1\\c_2\\a\\b\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1\\-2\end{bmatrix}$$

→ IVP solution