Eigenvector solution of a linear homogeneous system of differential equations

3x3 matrix of coefficients example

(scalar form)

$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \qquad x_1(0) = x_{10}$

$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \qquad x_2(0) = x_{20}$

$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \qquad x_3(0) = x_{30}$

OR

(explicit matrix form)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix}$$

$f' = \frac{df}{dt}$, $\vec{x}$ column matrix (distinguish scalar variables from vector variables!) OR $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ (symbolic matrix form) [or $\vec{x}' = A\vec{x}$ NOT $x' = Ax$!]

side calculation: find eigenvalues/eigenvectors of A: $\quad A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$

$\rightarrow (A - \lambda I)\vec{x} = \vec{0} \rightarrow |A - \lambda I| = 0 \rightarrow$ roots: $\lambda = \lambda_1, \lambda_2, \lambda_3$ (at least 1 real root!)

$\rightarrow (A - \lambda_i I)\vec{x} = \vec{0} \xrightarrow[\text{RREF}]{}$ L, F solve $\rightarrow$ identify basis vector(s) of each solution space:

$$\left.\begin{matrix} \lambda = \lambda_1, \lambda_2, \lambda_3 \\ B = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle \end{matrix}\right\} \quad A_B = B^{-1}AB = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ (diagonal)}$$

"diagonalization of matrix A"

otherwise A is "defective": method fails

linear change of variable (decoupling step):

$$\vec{x} = B\vec{y}, \quad \vec{y} = B^{-1}\vec{x}, \quad \vec{y}' = B^{-1}\vec{x}' = B^{-1}(A\vec{x}) = B^{-1}A(B\vec{y}) = A_B\vec{y}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \lambda_3 y_3 \end{bmatrix} \rightarrow \begin{matrix} y_1' = \lambda_1 y_1 \\ y_2' = \lambda_2 y_2 \\ y_3' = \lambda_3 y_3 \end{matrix} \rightarrow \begin{matrix} y_1 = c_1 e^{\lambda_1 t} \\ y_2 = c_2 e^{\lambda_2 t} \\ y_3 = c_3 e^{\lambda_3 t} \end{matrix}$$

(decoupled variables always have exponential solutions)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \langle \vec{b}_1 | \vec{b}_2 | \vec{b}_3 \rangle \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{bmatrix} = \underbrace{c_1 e^{\lambda_1 t}\vec{b}_1}_{\text{mode 1}} + \underbrace{c_2 e^{\lambda_2 t}\vec{b}_2}_{\text{mode 2}} + \underbrace{c_3 e^{\lambda_3 t}\vec{b}_3}_{\text{mode 3}}$$

general soln is an arbitrary linear combination of products of eigenvalue rate exponentials and corresponding eigenvectors

In each mode, the eigenvector fixes the ratios of the variables x_1, x_2, x_3 while their overall scale grows/decays exponentially; the eigenvalue is the rate factor.

initial conditions:

$$B\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = B^{-1}\begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix},$$

backsub values into gen soln, simplify x_1, x_2, x_3 combining modes

complex conjugate eigenvalue pairs

all this continues to work BUT one can combine the complex mode pair into a real pair of modes by evaluating the real and imaginary parts of either mode and using them instead, say mode 1, mode 2:

$$e^{\lambda_1 t}\vec{b}_1 = \vec{X}_1(t) + i\vec{X}_2(t), \quad \text{replace } c_1 e^{\lambda_1 t}\vec{b}_1 + c_2 e^{\lambda_2 t}\vec{b}_2 \text{ complex ☹}$$

$$\text{by } c_1\vec{X}_1(t) + c_2\vec{X}_2(t) \text{ real! ☺}$$

$$e^{(\pm k \pm i\omega)t} = \underbrace{e^{\pm kt}}_{\substack{\text{Re}(\lambda_1) \\ \text{governs the} \\ \text{real exponential}}}(\cos\omega t \pm i\sin\omega t)$$

$\uparrow$ $\text{Im}(\lambda)$ is the frequency for the sinusoidal factor

[polar form of eigenvector $\vec{b}_1$ gives relative ratios of amplitudes of x_1, x_2, x_3 in mode, while arguments give relative phase shifts]

polar form: $z = \underbrace{|z|}_{\text{magnitude}} e^{i\,\arg(z)}$ — argument = radian angle