DE pictures: sinusoidal example

$y'' + y = 0$

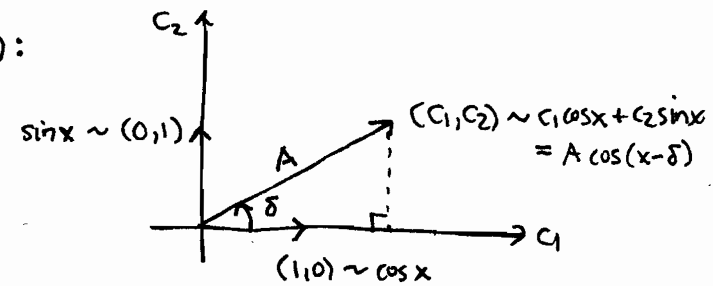
solutions : $y = \overbrace{c_1 \cos x + c_2 \sin x}^{\text{"sinusoidal function"}} \longleftrightarrow (c_1, c_2)$:

$= A \cos(x - \delta)$

amplitude, unit frequency, phase shift $\in (-\pi, \pi]$

"sinusoidal functions" are phase shifted cosines or sines with an amplitude coefficient

SOLN SPACE PICTURE



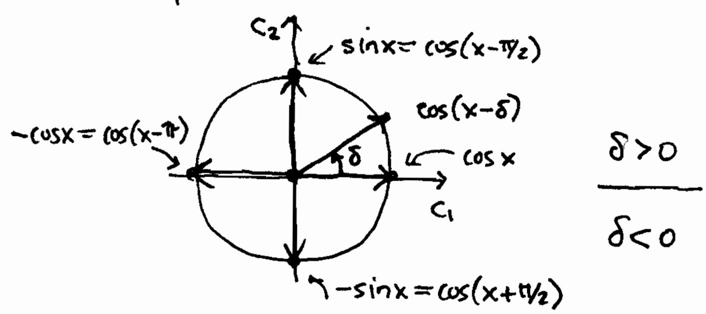
$\{y_1, y_2\} = \{\cos x, \sin x\}$ basis of soln space corresponds to basis $\{\hat{\imath} = (1,0), \hat{\jmath} = (0,1)\}$ of coefficient plane.

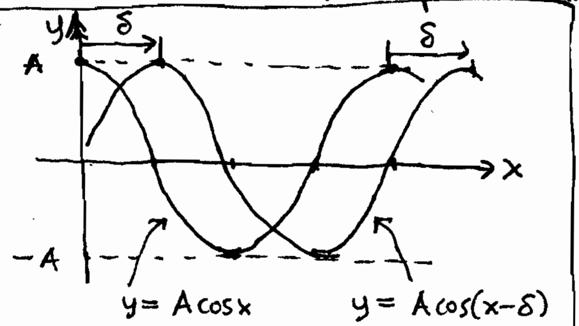
each soln corresponds to a 2-vector.

(c_1, c_2) like Cartesian coords (x, y).

(A, δ) like polar coords (r, θ). [details later]

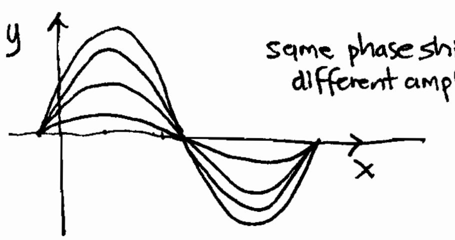
The unit circle $A^2 = c_1^2 + c_2^2 = 1$ corresponds to all unit amplitude solutions:





$\delta > 0$ as shown: peaks occur at later value of x

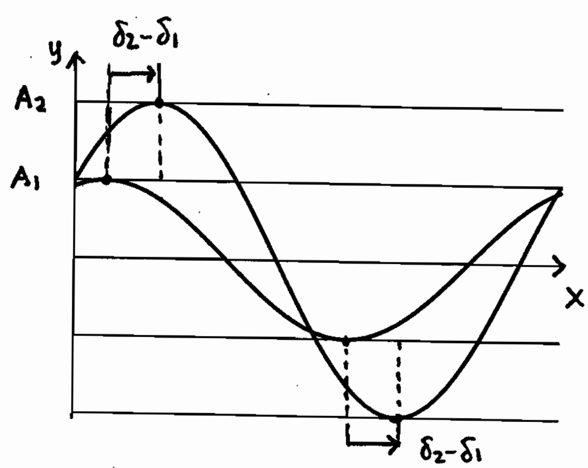
same amplitude different phase shift.

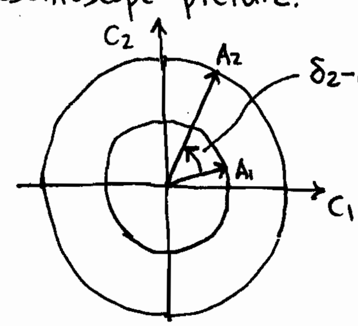


same phase shift different amplitude

OSCILLOSCOPE PICTURE

Plotting solns in the soln space picture visually shows relative amplitude and phase which characterizes what you see in the oscilloscope picture.





$\delta_2 - \delta_1 > 0$ as shown

$\frac{A_2}{A_1} > 1$ as shown

Note that if x is a time variable, the curve with peaks at larger x values is behind in time (later time)

In this sense the sine lags behind the cosine by $90°$: $\delta = \pi/2 > 0$.

"phase" is just another word for the input angle for a trig function:

$\cos(\underbrace{\ldots}_{\text{phase}})$

DE pictures: sinusoidal example (2): initial data problem

$$y(x) = c_1 \cos x + c_2 \sin x \quad = c_1 y_1 + c_2 y_2$$
$$y'(x) = c_1(-\sin x) + c_2 \cos x \quad = c_1 y_1' + c_2 y_2'$$

$$y_0 = y(x_0) = c_1 \cos x_0 + c_2 \sin x_0 \quad = c_1 y_1(x_0) + c_2 y_2(x_0)$$
$$v_0 = y'(x_0) = c_1(-\sin x_0) + c_2 \cos x_0 \quad = c_1 y_1'(x_0) + c_2 y_2'(x_0)$$

matrix form:
$$\begin{bmatrix} \cos x_0 & \sin x_0 \\ -\sin x_0 & \cos x_0 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{to find}} = \underbrace{\begin{bmatrix} y_0 \\ v_0 \end{bmatrix}}_{\text{given}} \qquad \underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{W(y_1,y_2)(x_0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$$

Wronskian matrix for basis $\{y_1, y_2\}$ evaluated at $x = x_0$

at $x_0 = 0$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} \qquad \begin{matrix} c_1 = y_0 \\ c_2 = v_0 \end{matrix} \qquad \text{note: } W(y_1,y_2)(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The coefficients are directly the initial data parameters at $x_0 = 0$ where the Wronskian matrix is the identity. The basis $\{y_1, y_2\} = \{\cos x, \sin x\}$ is "natural" for initial conditions at $x_0 = 0$:

$$y = y_0 \cos x + v_0 \sin x$$

at $x_0 \neq 0$: Solve by multiplying by the inverse: $\begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = W(y_1,y_2)(x_0)^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \cos x_0 & \sin x_0 \\ -\sin x_0 & \cos x_0 \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \frac{1}{\cos^2 x_0 + \sin^2 x_0} \begin{bmatrix} \cos x_0 & -\sin x_0 \\ \sin x_0 & \cos x_0 \end{bmatrix} \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} y_0 \cos x_0 - v_0 \sin x_0 \\ y_0 \sin x_0 + v_0 \cos x_0 \end{bmatrix}$$

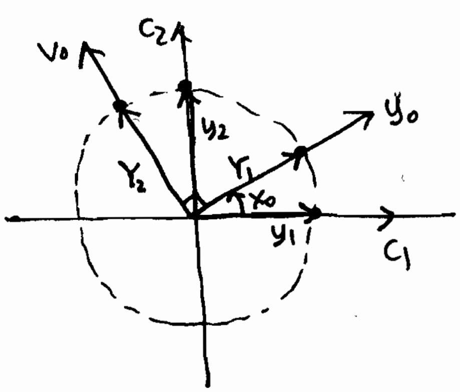
backsub:
$$y = (y_0 \cos x_0 - v_0 \sin x_0) \cos x + (y_0 \sin x_0 + v_0 \cos x_0) \sin x$$
$$= y_0(\cos x \cos x_0 + \sin x \sin x_0) + v_0(\sin x \cos x_0 - \cos x \cos x_0) \quad \text{use subtraction identities}$$
$$= y_0 \underbrace{\cos(x - x_0)}_{Y_1} + v_0 \underbrace{\sin(x - x_0)}_{Y_2}$$

Solving the initial data conditions is equivalent to finding a new basis $\{Y_1, Y_2\}$ of the solution space for which the Wronskian matrix evaluated at x_0 is the identity matrix:

$$W(Y_1,Y_2)(x) = \begin{bmatrix} \cos(x - x_0) & \sin(x - x_0) \\ -\sin(x - x_0) & \cos(x - x_0) \end{bmatrix} \qquad W(Y_1,Y_2)(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The new basis $\{Y_1, Y_2\}$ is "natural" for initial conditions at $x = x_0$.
[Note that these are exactly the translations of $\{y_1, y_2\}$ from $x = 0$ to $x = x_0$.]

The corresponding new coordinates are the initial data parameters (y_0, v_0) for initial data at $x = x_0$. In this case they are just rotated by an angle x_0 from the initial coordinates (c_1, c_2).



$$\underset{\text{old}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = B \underset{\text{new}}{\begin{bmatrix} y_0 \\ v_0 \end{bmatrix}} \qquad \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \underbrace{B^{-1}}_{W(y_1,y_2)(x_0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The basis changing matrix is the inverse of the Wronskian matrix of the old basis evaluated at the initial data point $x = x_0$.