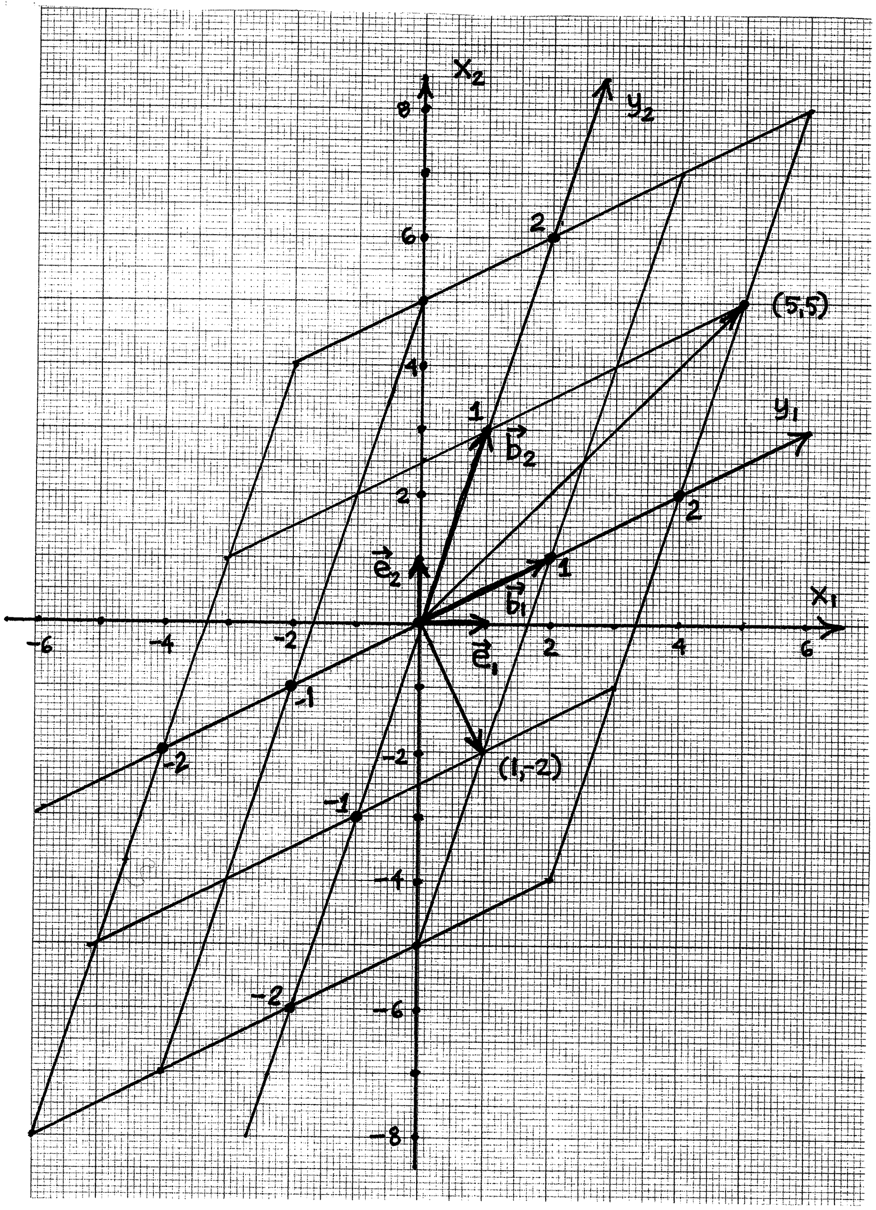
Nonstandard coordinates on $\mathbb{R}^2$



- old basis:
  $\vec{e}_1 = \langle 1,0\rangle$, $\vec{e}_2 = \langle 0,1\rangle$

  new basis:
  $\vec{b}_1 = \langle 2,1\rangle$, $\vec{b}_2 = \langle 1,3\rangle$

basis changing matrix:

$B = \text{augment}(\vec{b}_1, \vec{b}_2) = \langle \vec{b}_1 | \vec{b}_2 \rangle$

$= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ cols = old coords of new basis vectors

inverse:

$B^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$

cols = new coords of old basis vectors

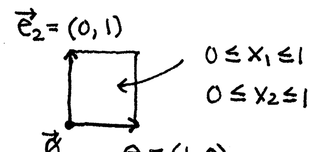
2 example vectors:

$\langle 5,5\rangle = 5\vec{e}_1 + 5\vec{e}_2$
$\quad = 2\vec{b}_1 + 1 b_2$

$x_1 = 5,\ x_2 = 5$
$y_1 = 2,\ y_2 = 1$

$\langle 1,-2\rangle = 1\vec{e}_1 - 2\vec{e}_2$
$\quad = 1\vec{b}_1 - 1\vec{b}_2$

$x_1 = 1,\ x_2 = -2$
$y_1 = 1,\ y_2 = -1$



The graph paper has the unit coordinate grid for the standard cartesian coordinates $\{x_1, x_2\}$ marked by bold line 1cm tickmarks (refined to 1mm tenth tickmarks).

The unit coordinate rectangle at the origin tiles the plane to make this grid.

The new unit coordinate grid is a tiling of the plane by the unit coordinate "rectangle" $0 \le y_1 \le 1$, $0 \le y_2 \le 1$ (actually a parallelogram) at the origin. The region shown is $-2 \le y_1 \le 2$, $-2 \le y_2 \le 2$.

The relationship between the two coordinate systems

$$\vec{X} = \langle x_1, x_2\rangle = x_1\vec{e}_1 + x_2\vec{e}_2 = y_1\vec{b}_1 + y_2\vec{b}_2 = y_1\langle 2,1\rangle + y_2\langle 1,3\rangle$$

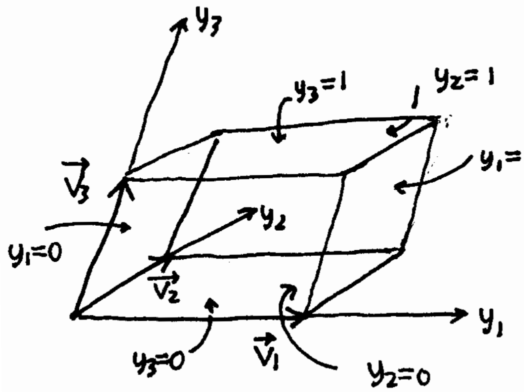
or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\underset{\text{"old"}}{X} = B\underset{\text{"new"}}{Y} \longleftrightarrow \underset{\text{"new"}}{Y} = B^{-1}\underset{\text{"old"}}{X}$

or $x_1 = 2y_1 + y_2$, $x_2 = 1y_1 + 3y_2$ — linear change of coordinates — $y_1 = (3x_1 - x_2)/5$, $y_2 = (-x_1 + 2x_2)/5$

Nonstandard Coordinates in $\mathbb{R}^3$

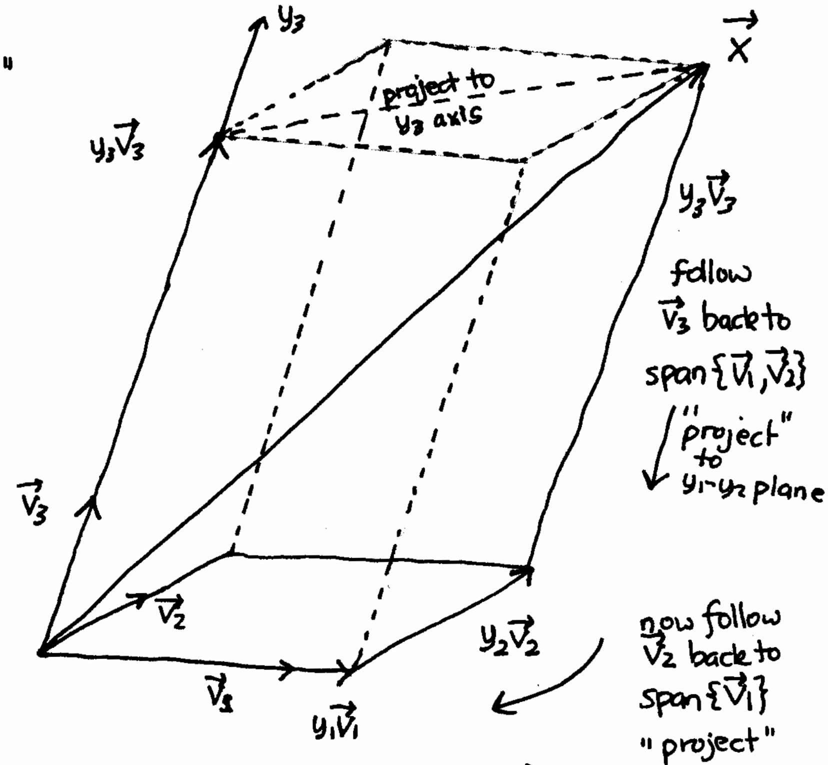
basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$ "new basis"
= any 3 LI vectors



y_i coord lines are parallel to vector $\vec{v}_i$

y_i coord planes are parallel to planes of remaining 2 vectors

This parallelopiped above represents all points $\vec{x}$ for which:
$0 \le y_1 \le 1, \quad 0 \le y_2 \le 1, \quad 0 \le y_3 \le 1.$



$$\vec{x} = ((y_1\vec{v}_1) + y_2\vec{v}_2) + y_3\vec{v}_3$$

first $\leftarrow$ to $\quad$ last

- GEOMETRICAL DETERMINATION OF NEW COORDINATES OF A VECTOR $\vec{x}$

- MATRIX DETERMINATION OF NEW COORDINATES OF A VECTOR $\vec{x}$

Augment basis vectors into columns of a matrix (basis changing matrix)

$B = \text{augment}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$
$$\quad\ \uparrow \qquad \uparrow \qquad \uparrow$$
$$\quad\ \vec{v}_1 \qquad \vec{v}_2 \qquad \vec{v}_3$$

each column consists of the old (standard) coordinates of a new basis vector

Now express any vector (standard coordinates) in terms of the new basis:

$$\underset{\text{"old coords"}}{\vec{x}} = y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 = B\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = B\underset{\text{"new coords"}}{\vec{y}}$$

Solve $\vec{x} = B\vec{y}$ to find new coords $\vec{y}$ of $\vec{x}$.

LI of new basis vectors guarantees that if $\vec{x} = 0$ then $\vec{y} = 0$ is only solution, so $\det(B) \neq 0$ and B^{-1} exists. Solution equivalent to: $\underset{\text{"new coords"}}{\vec{y}} = B^{-1}\underset{\text{"old coords"}}{\vec{x}}$
unique!