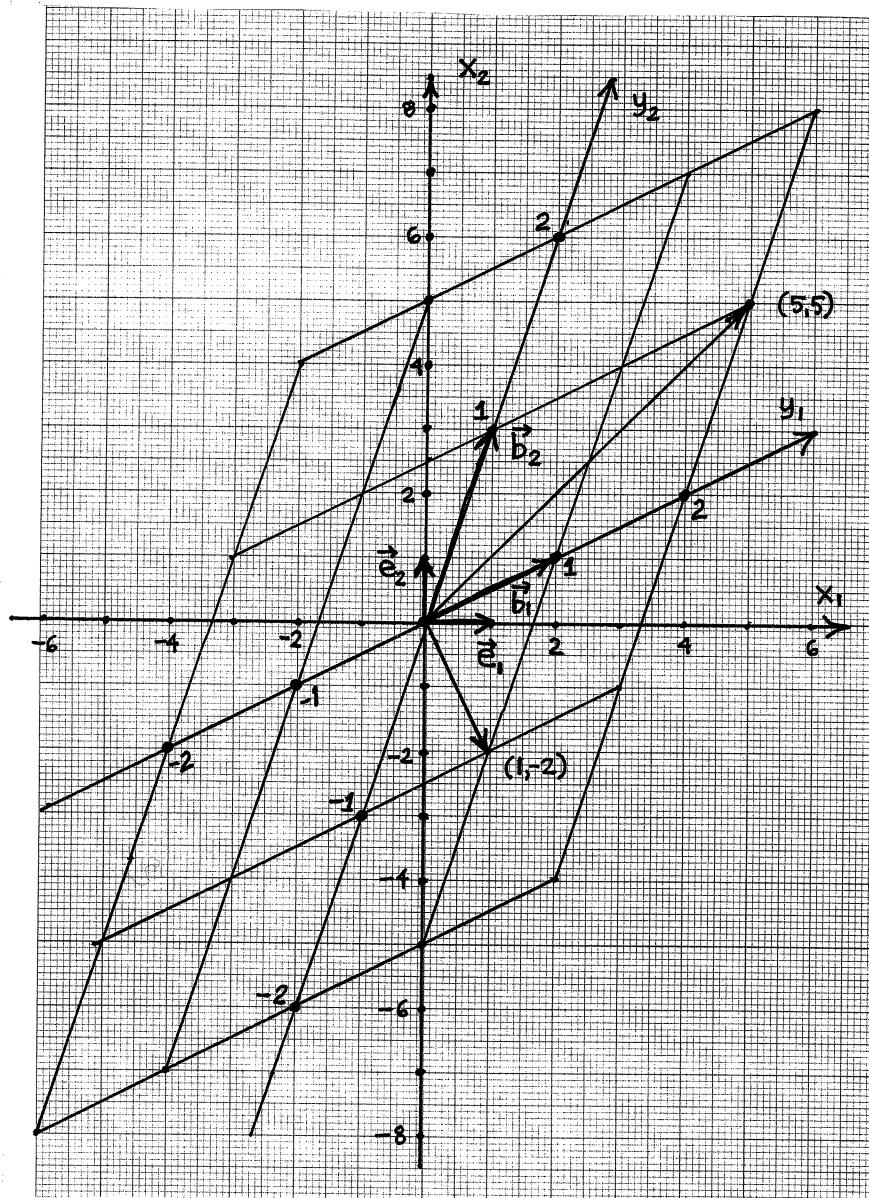


Nonstandard coordinates on \mathbb{R}^2



old basis:
 $\vec{e}_1 = \langle 1, 0 \rangle, \vec{e}_2 = \langle 0, 1 \rangle$
 new basis:
 $\vec{b}_1 = \langle 2, 1 \rangle, \vec{b}_2 = \langle 1, 3 \rangle$

basis changing matrix:
 $B = \text{augment } (\vec{b}_1, \vec{b}_2) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$
 $= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ cols = old coords of new basis vectors

inverse:
 $B^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$
 cols = new coords of old basis vectors

2 example vectors:

$$\begin{aligned} \langle 5, 5 \rangle &= 5\vec{e}_1 + 5\vec{e}_2 \\ &= 2\vec{b}_1 + 1\vec{b}_2 \end{aligned}$$

$$\begin{aligned} x_1 &= 5, x_2 = 5 \\ y_1 &= 2, y_2 = 1 \end{aligned}$$

$$\begin{aligned} \langle 1, -2 \rangle &= 1\vec{e}_1 - 2\vec{e}_2 \\ &= 1\vec{b}_1 - 1\vec{b}_2 \end{aligned}$$

$$\begin{aligned} x_1 &= 1, x_2 = -2 \\ y_1 &= 1, y_2 = -1 \end{aligned}$$

$$\begin{array}{l} \vec{e}_2 = (0, 1) \\ \vec{e}_1 = (1, 0) \\ \text{unit} \end{array}$$

The graph paper has the unit coordinate grid for the standard cartesian coordinates $\{x_1, x_2\}$ marked by bold line 1cm tickmarks (refined to 1mm tenth tickmarks).

The unit coordinate rectangle at the origin tiles the plane to make this grid.

The new coordinate grid is a tiling of the plane by the unit coordinate "rectangle" $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$ (actually a parallelogram) at the origin. The region shown is $-2 \leq y_1 \leq 2, -2 \leq y_2 \leq 2$.

The relationship between the two coordinate systems

$$\vec{X} = \langle x_1, x_2 \rangle = x_1 \vec{e}_1 + x_2 \vec{e}_2 = y_1 \vec{b}_1 + y_2 \vec{b}_2 = y_1 \langle 2, 1 \rangle + y_2 \langle 1, 3 \rangle$$

$$\text{or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{or } \begin{matrix} \vec{X} = B \vec{Y} \\ \text{"old"} \quad \text{"new"} \end{matrix} \leftrightarrow \begin{matrix} \vec{Y} = B^{-1} \vec{X} \\ \text{"new"} \quad \text{"old"} \end{matrix}$$

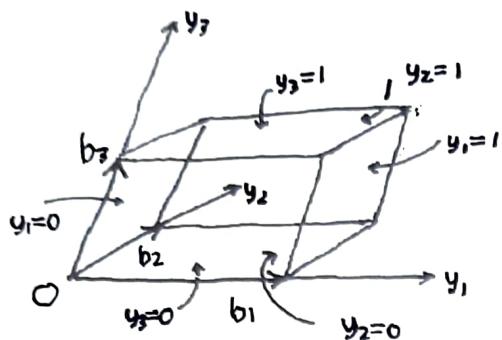
$$\text{or } \begin{aligned} x_1 &= 2y_1 + y_2 \\ x_2 &= y_1 + 3y_2 \end{aligned}$$

linear change
of coordinates

$$\begin{aligned} y_1 &= (3x_1 - x_2)/5 \\ y_2 &= (-x_1 + 2x_2)/5 \end{aligned}$$

Nonstandard Coordinates in \mathbb{R}^3

basis $\{b_1, b_2, b_3\}$ of \mathbb{R}^3
= any 3 LI vectors
"new basis"

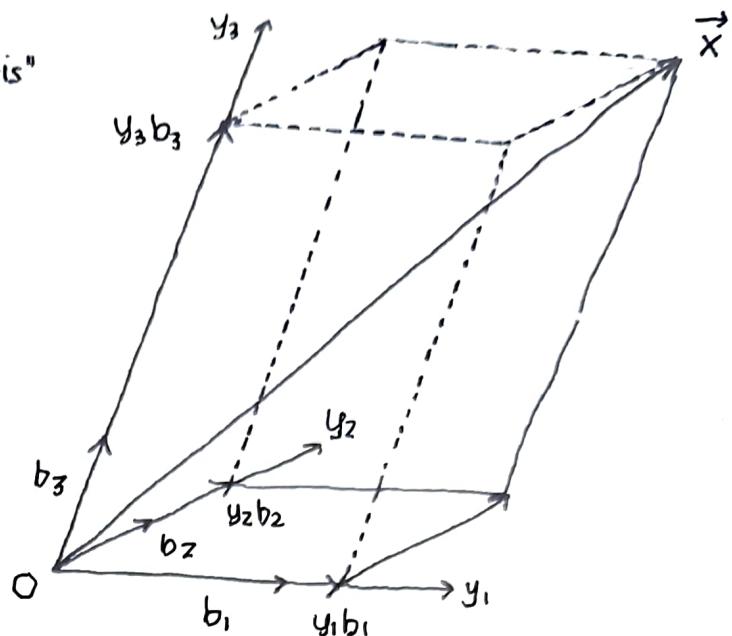


y_i coord lines are parallel to vector b_i

y_i coord planes are parallel to planes of remaining 2 vectors

This parallelopiped above represents all points \vec{x} for which:

$$0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_3 \leq 1$$



$$\vec{x} = y_1 b_1 + y_2 b_2 + y_3 b_3$$

The vector sum of the three edges coming out of the origin equals the target vector \vec{x} .

This shows the case $y_1 > 0$
 $y_2 > 0$
 $y_3 > 0$.

Matrix version

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y_1 b_1 + y_2 b_2 + y_3 b_3 = \langle b_1 | b_2 | b_3 \rangle \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

old coords new coords

columns coefficients

B

Given new coords,
calculate
old coords

$$X = By$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = B^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

new coords old coords

Given old coords,
calculate
new coords

$$Y = B^{-1} X$$