Complex coordinates in the plane: an example

$$A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \xrightarrow{\text{Maple}} \quad \lambda = -1+2i \quad -1-2i$$

$$B = \begin{bmatrix} -\frac{1}{2}-\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} \\ 1 & 1 \end{bmatrix} \longrightarrow B = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}$$

scale up for easier algebra:

confirm A is diagonalized: $A_B = B^{-1} A B = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$

complex change of variables:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 b_1 + y_2 b_2 = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2i & 1+i \\ -2i & 1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

old point: $\langle x_1, x_2 \rangle = \langle 1, 2 \rangle$

new coords: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2i & 1+i \\ -2i & 1-i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2i + 2 + 2i \\ -2i + 2 - 2i \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + i \\ \frac{1}{2} - i \end{bmatrix}$

Now evaluate the product with the complex eigenvalue exponential:

$$e^{\lambda_1 t} = e^{(-1+2i)t} = e^{-t} e^{2it} = e^{-t}(\cos 2t + i \sin 2t) \equiv e^{-t}(c + is)$$

abbreviation convenient

$$e^{\lambda_1 t} b_1 = e^{-t}(c+is) \begin{bmatrix} -1-i \\ 2 \end{bmatrix} = e^{-t} \begin{bmatrix} (-c+s) + i(-c-s) \\ 2c + i(2s) \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} -c+s \\ 2c \end{bmatrix} + i\, e^{-t} \begin{bmatrix} -c-s \\ 2s \end{bmatrix} \equiv X_1 + i X_2$$

$$y_1 e^{\lambda_1 t} b_1 = \left(\tfrac{1}{2} + i\right)(X_1 + i X_2) = \left(\tfrac{1}{2} X_1 - X_2\right) + i\left(X_1 + \tfrac{1}{2} X_2\right)$$

$$= \left[\frac{1}{2}\begin{bmatrix} -c+s \\ 2c \end{bmatrix} - \begin{bmatrix} -c-s \\ 2s \end{bmatrix} + i\left(\begin{bmatrix} -c+s \\ 2c \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -c-s \\ 2s \end{bmatrix}\right)\right] e^{-t}$$

$$= e^{-t} \begin{bmatrix} \frac{1}{2}c + \frac{3}{2}s \\ c - 2s \end{bmatrix} + i\, e^{-t} \begin{bmatrix} -\frac{3}{2}c + \frac{1}{2}s \\ 2c + s \end{bmatrix}$$

$$x(t) = y_1 e^{\lambda_1 t} b_1 + y_2 e^{\lambda_2 t} b_2 = 2\,\mathrm{Re}(y_1 e^{\lambda_1 t} b_1) = e^{-t} \begin{bmatrix} \cos 2t + 3 \sin 2t \\ 2\cos 2t - 4 \sin 2t \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

we will see how this is a solution of: $x' = Ax$, $x(0) = \langle 1, 2 \rangle$.

$$x' = y_1 \lambda_1 e^{\lambda_1 t} b_1 + y_2 \lambda_2 e^{\lambda_2 t} b_2$$

$$Ax = y_1 e^{\lambda_1 t} A b_1 + y_2 e^{\lambda_2 t} A b_2 = y_1 e^{\lambda_1 t} \lambda_1 b_1 + y_2 e^{\lambda_2 t} \lambda_2 b_2 \quad !!$$