From a complex to a real basis of solutions

Suppose $z(t)$ and $\bar{z}(t)$ are complex solutions of a real linear homogeneous differential equation.
Namely $z(t) = x(t) + i\,y(t)$, $\bar{z}(t) = x(t) - i\,y(t)$ which contain only 2 real functions.

change of variables:

$$z = x + iy$$
$$\bar{z} = x - iy$$

sum: $z + \bar{z} = 2x \rightarrow x = \frac{1}{2}(z+\bar{z}) = \mathrm{Re}(z) = \mathrm{Re}(\bar{z})$

diff: $z - \bar{z} = 2iy \rightarrow y = \frac{1}{2i}(z-\bar{z}) = \mathrm{Im}(z) = -\mathrm{Im}(\bar{z})$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{bmatrix}}_{B} \begin{bmatrix} z \\ \bar{z} \end{bmatrix}$$ linear change of coordinates

$B \rightarrow \det B = -\frac{1}{2i} \neq 0$ so x and y are linearly independent combinations of z and $\bar{z}$

Thus any linear combination of $z, \bar{z}$ can be expressed as a linear combination of x, y and vice versa.

CONCLUSION:
If a soln technique produces solutions $z(t)$, $\bar{z}(t)$ so that by linearity $C_1 z(t) + C_2 \bar{z}(t)$ is also a solution, then instead one can use $c_1 x(t) + c_2 y(t)$. If c_1, c_2 are real, we get an explicitly real solution formula.

EX. $e^{(a+ib)t}, e^{(a-ib)t} \longrightarrow e^{at}\cos bt, e^{at}\sin bt$ (Re and Im parts of positive imaginary part exponential)

EX $e^{(a+ib)t}\vec{b}_1, e^{(a-ib)t}\vec{B}_1 \rightarrow \mathrm{Re}(e^{(a+ib)t}\vec{b}_1), \mathrm{Im}(e^{(a+ib)t}\vec{b}_1)$

suppose $\vec{b}_1 = \vec{u} + i\vec{v}$

$$e^{(a+ib)t} b_1 = e^{at}(\cos bt + i \sin bt)(\vec{u} + i\vec{v})$$
$$= \underbrace{e^{at}(\cos bt\,\vec{u} - \sin bt\,\vec{v})}_{\substack{\text{Re part} \\ \vec{B}_1(t)}} + i\,\underbrace{e^{at}(\sin bt\,\vec{u} + \cos bt\,\vec{v})}_{\substack{\text{Im part} \\ \vec{B}_2(t)}}$$

real soln : $c_1\vec{B}_1(t) + c_2\vec{B}_2(t)$

The only difference is that now we are dealing with complex vector solutions of a real linear homogeneous vector differential equation.