

complex number arithmetic

$$z = \underbrace{x}_{\text{Re}(z)} + \underbrace{iy}_{\text{Im}(z)}$$

$$i = \sqrt{-1}, i^2 = -1$$

multiplication:

$$\begin{aligned} z_3 &= z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + \cancel{i^2 y_1 y_2} + i(x_1 y_2 + y_1 x_2) \\ &= \underbrace{(x_1 x_2 - y_1 y_2)}_{x_3} + i \underbrace{(x_1 y_2 + y_1 x_2)}_{y_3} \end{aligned}$$

division:

$$\begin{aligned} z_3 &= z_1 / z_2 = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_2 - x_1 y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_2 x_2 - x_1 y_2}{x_2^2 + y_2^2} \end{aligned}$$

Definition

$$e^{i\theta} = \cos\theta + i\sin\theta$$

, θ real

complex exponential

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)} = e^{i\theta_1 + i\theta_2} \end{aligned}$$

exp sum rule ✓

$$e^0 = \cos 0 + i\sin 0 = 1 + i0 = 1 \quad \checkmark$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \overline{(\cos\theta + i\sin\theta)} = \overline{e^{i\theta}} \quad \text{complex conjugation} \checkmark$$

$$|e^{i\theta}|^2 = e^{i\theta} \overline{e^{i\theta}} = e^{i\theta} e^{-i\theta} = e^0 = 1 \rightarrow \text{"unit complex number," lies on unit circle in complex plane at polar angle } \theta.$$

$$(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta) = \cos 2\theta + i\sin 2\theta = e^{i2\theta} \quad \text{power rule for exponents} \checkmark$$

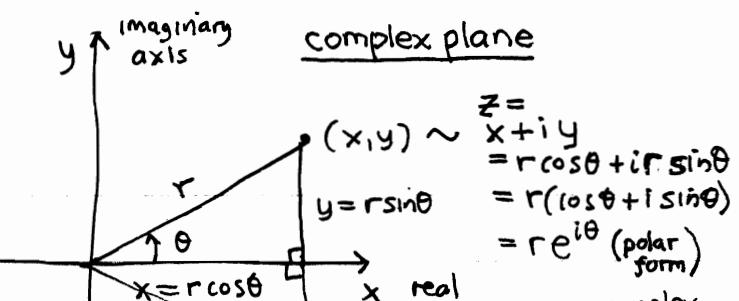
$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta} \quad \text{polar form}$$

$$\begin{aligned} z_1 z_2 &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\ z_1/z_2 &= (r_1 e^{i\theta_1}) / (r_2 e^{i\theta_2}) = (r_1/r_2) e^{i(\theta_1 - \theta_2)} \end{aligned} \quad \left. \begin{array}{l} \times, \div \text{ easy in polar form.} \end{array} \right\}$$

$$\begin{aligned} \text{Calculus: } \frac{d}{dx} e^{ix} &= \frac{d}{dx} (\cos x + i\sin x) = \frac{d}{dx} (\cos x) + i \frac{d}{dx} (\sin x) = -\sin x + i\cos x \\ &= i(\cos x + i\sin x) = ie^{ix} \quad \checkmark \text{ just like for real coeff's} \end{aligned}$$

$$\int e^{ix} dx = \frac{1}{i} e^{ix} + C$$

$$e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax} (\cos bx + i\sin bx)$$



complex plane

$$\begin{aligned} z &\sim x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \quad (\text{polar form}) \\ (x, -y) &\sim \overline{z} \leftarrow \\ &= x - iy \end{aligned}$$

$$|z|^2 = z \overline{z} = (x+iy)(x-iy) = x^2 - i^2 y^2 = x^2 + y^2$$

$$|z| = \sqrt{x^2 + y^2} = r$$

↑ absolute value

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$\theta = \arg(z)$ "argument"

("principal argument" $\text{Arg}(z)$)
if choose $-\pi < \theta \leq \pi$.

complex exponential

exp sum rule ✓

complex conjugation ✓

"unit complex number," lies on unit circle in complex plane at polar angle θ .

power rule for exponents ✓

$$\begin{cases} \frac{d}{dx} e^{(a+ib)x} = (a+ib)e^{(a+ib)x} \\ \int e^{(a+ib)x} dx = \frac{e^{(a+ib)x}}{a+ib} + C \end{cases}$$

complex constant

complex number arithmetic (2)

- Taylor series defines complex exponential. Purely imaginary case:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \underbrace{\frac{i^n x^n}{n!}}_{\substack{\text{split: even } n=2k \\ \text{odd } n=2k+1}} = \sum_{k=0}^{\infty} \underbrace{\frac{i^{2k} x^{2k}}{(2k)!}}_{(-1)^k} + \sum_{k=0}^{\infty} \underbrace{\frac{i^{2k+1} x^{2k+1}}{(2k+1)!}}_{(-1)^k i}$$

$$= \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \right) + i \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right) = \cos x + i \sin x \quad \checkmark$$

extend by exponent rule: $e^{(a+ib)x} = e^{ax+ibx} = e^{ax} e^{ibx} = e^{ax} (\cos bx + i \sin bx)$

- $\text{span}\{e^{ix}, e^{-ix}\} = \text{span}\{\cos x, \sin x\}$:

$$e^{ix} = \cos x + i \sin x \rightarrow \text{sum}/2 \rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$e^{-ix} = \cos x - i \sin x \rightarrow \text{diff}/2i \rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

any linear combination of $\{e^{ix}, e^{-ix}\}$ can be expressed in terms of $\{\cos x, \sin x\}$ and vice versa

- $\text{span}\{e^{(a+ib)x}, e^{(a-ib)x}\} = \text{span}\{e^{ax} \cos bx, e^{ax} \sin bx\}$

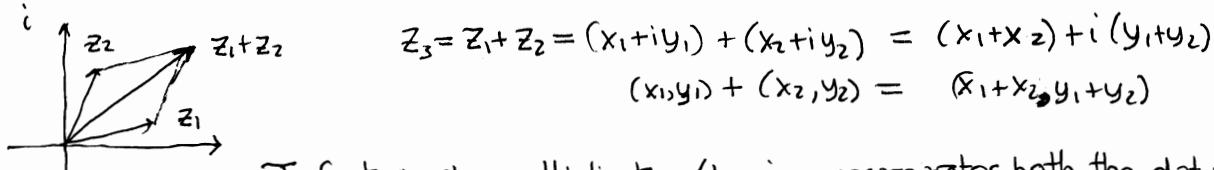
$$e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax} (\cos bx + i \sin bx) \longrightarrow e^{ax} \cos bx = \frac{e^{(a+ib)x} + e^{(a-ib)x}}{2}$$

$$e^{(a-ib)x} = e^{ax} e^{-ibx} = e^{ax} (\cos bx - i \sin bx) \longrightarrow e^{ax} \sin bx = \frac{e^{(a+ib)x} - e^{(a-ib)x}}{2i}$$

any linear combination of $\{e^{(a+ib)x}, e^{(a-ib)x}\}$ can be expressed in terms of $\{e^{ax} \cos bx, e^{ax} \sin bx\}$ and vice versa

afterthought

obvious: complex addition/subtraction corresponds directly to vector addition/subtraction in \mathbb{R}^2 thought of as the "complex plane"



In fact complex multiplication/division incorporates both the dot product and cross products together; if we let $\vec{r} = (x, y, 0)$ to evaluate cross-products, then

$$\begin{aligned} \bar{z}_1 z_2 &= (x_1 - iy_1)(x_2 + iy_2) = (x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1) \\ &= \vec{r}_1 \cdot \vec{r}_2 + i \hat{k} \cdot (\vec{r}_1 \times \vec{r}_2) \end{aligned}$$

$$\hat{k} = (0, 0, 1)$$