Damped Harmonic Oscillator Driven by Sinusoidal Driving Function: Special Cases

BEATING: drive undamped system $(k_0 = 0)$ near the natural frequency : $\omega \approx \omega_0$

Take initial conditions $y(0) = 0 = y'(0)$ (nothing happening at $t=0$)

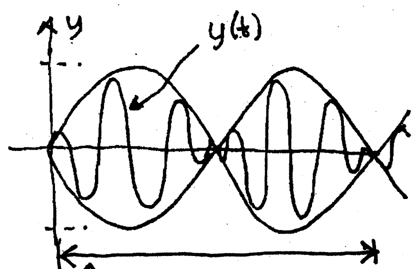
The general formula reduces to

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{B_0}{\omega_0^2-\omega^2}\cos\omega t \qquad y(0) = c_1 + \frac{B_0}{\omega_0^2-\omega^2} = 0 \to c_1 = \frac{-B_0}{\omega_0^2-\omega^2}$$

$$y' = -c_1\omega_0 \sin\omega_0 t + c_2\omega_0\cos\omega_0 t - \frac{B_0\omega}{\omega_0^2-\omega^2} \qquad y'(0) = c_2\omega_0 = 0 \to c_2 = 0$$

$$y = \frac{B_0}{\omega_0^2-\omega^2}\left(-\cos\omega_0 t + \cos\omega t\right) = \underbrace{\left[\frac{+2B_0}{\omega_0^2-\omega^2}\underbrace{\sin\left(\tfrac{\omega_0-\omega}{2}t\right)}_{\substack{\text{smaller frequency,}\\ \text{larger period of}\\ \text{oscillation}}}\right]}_{\substack{A(t)\ \text{sinusoidal amplitude for larger}\\ \text{frequency oscillation}}} \underbrace{\sin\left(\tfrac{\omega_0+\omega}{2}t\right)}_{\substack{\text{larger frequency,}\\ \text{smaller period of}\\ \text{oscillation}}}$$

trig difference identity



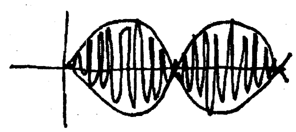
$y = \pm A(t)$ (envelope)

2 beats: $T = \dfrac{2\pi}{\frac{|\omega_0-\omega|}{2}}$

beating is the constructive/destructive interference of two sinusoidal signals/waves with nearly the same frequency and intensity (take 2 tuning forks for nearby musical notes)

if $\omega \approx \omega_0$, then $\left|\frac{\omega_0-\omega}{2}\right| \ll \frac{\omega_0+\omega}{2}$

so it looks more like



RESONANCE if $k_0 = 0$ (no damping) the above formulas are not valid for $\omega = \omega_0$ so driving the undamped system at its natural frequency must be handled separately.

$(D^2+\omega_0^2)y = B_0\cos\omega_0 t \leftarrow (D^2+\omega_0^2)(B_0\cos\omega_0 t) = 0$ same roots $r = \pm i\omega_0$

$(D^2+\omega_0^2)^2 y = 0$ $\qquad y = (c_1+c_3 t)\cos\omega_0 t + (c_2+c_4 t)\sin\omega_0 t$ (multiplicity 2 need linear function coefficients)

$(r^2+\omega_0^2)^2 = 0 \to r = \pm i\omega_0$ mult: 2

$$= \underbrace{(c_1\cos\omega_0 t + c_2\sin\omega_0 t)}_{y_h\ \text{(fixed amplitude)}} + \underbrace{t(c_3\cos\omega_0 t + c_4\sin\omega_0 t)}_{y_p\ \text{(growing amplitude)}}$$

$\omega_0^2\,[y_p = t(c_3\cos\omega_0 t + c_4\sin\omega_0 t)]$

$0\,[Dy_p = t(-c_3\omega_0\sin\omega_0 t + c_4\omega\cos\omega_0 t) + (c_3\cos\omega_0 t + c_4\sin\omega_0 t)]$

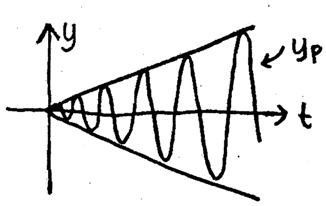
$1\,[D^2 y_p = t(-c_3\omega_0^2\cos\omega_0 t - c_4\omega_0^2\sin\omega_0 t) + 2(-c_3\omega_0\sin\omega_0 t + c_4\omega_0\cos\omega_0 t)]$

$D^2 y_p + \omega_0^2 y_p = t(0) + 2(-c_3\omega_0\sin\omega_0 t + c_4\omega_0\cos\omega_0 t) = B_0\cos\omega_0 t$

$\therefore \left.\begin{aligned} -2c_3\omega_0 = 0 &\to c_3 = 0 \\ 2c_4\omega_0 = B_0 &\to c_4 = B_0/(2\omega_0)\end{aligned}\right\}\ y_p = \underbrace{\left(\frac{B_0 t}{2\omega_0}\right)}_{A(t) = \frac{B_0 t}{2\omega_0}}\sin\omega_0 t = A(t)\cos\left(\omega_0 t - \frac{\pi}{2}\right)$

90° out of phase [y_p in phase with $f(t)$]

linearly increasing amplitude



oscillation grows until model breaks down (no damping assumption no longer justified, etc)