Damped Harmonic Oscillator Driven by Sinusoidal Driving Function: Special Cases

BEATING: drive undamped system ($k_0 = 0$) near the natural frequency: $\omega \approx \omega_0$

Take initial conditions $y(0) = 0 = y'(0)$ (nothing happening at $t=0$)

The general formula reduces to

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{B_0}{\omega_0^2-\omega^2}\cos\omega t \qquad y(0) = c_1 + \frac{B_0}{\omega_0^2-\omega^2} = 0 \to c_1 = \frac{-B_0}{\omega_0^2-\omega^2}$$

$$y' = -c_1\omega_0 \sin\omega_0 t + c_2\omega_0\cos\omega_0 t - \frac{B_0\omega}{\omega_0^2-\omega^2} \qquad y'(0) = c_2\omega_0 = 0 \to c_2 = 0$$

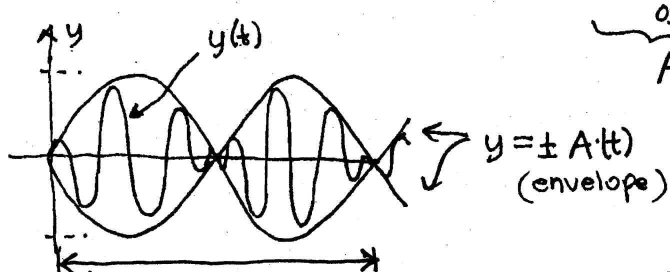
$$y = \frac{B_0}{\omega_0^2-\omega^2}(-\cos\omega_0 t + \cos\omega t) = \left[\frac{+2B_0}{\omega_0^2-\omega^2}\sin\left(\frac{\omega_0-\omega}{2}t\right)\right]\sin\left(\frac{\omega_0+\omega}{2}t\right)$$

trig difference identity

$\sin\left(\frac{\omega_0-\omega}{2}t\right)$: smaller frequency, larger period of oscillation

$\sin\left(\frac{\omega_0+\omega}{2}t\right)$: larger frequency, smaller period of oscillation

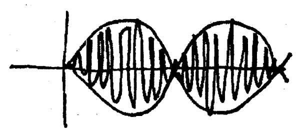
$A(t)$ sinusoidal amplitude for larger frequency oscillation



2beats: $T = \frac{2\pi}{\frac{|\omega_0-\omega|}{2}}$

beating is the constructive/destructive interference of two sinusoidal signals/waves with nearly the same frequency and intensity (take 2 tuning forks for nearby musical notes)

if $\omega \approx \omega_0$, then $\left|\frac{\omega_0-\omega}{2}\right| << \frac{\omega_0+\omega}{2}$ so it looks more like



RESONANCE if $k_0 = 0$ (no damping) the above formulas are not valid for $\omega = \omega_0$ so driving the undamped system at its natural frequency must be handled separately.

$(D^2+\omega_0^2)y = B_0\cos\omega_0 t \leftarrow (D^2+\omega_0^2)(B_0\cos\omega_0 t) = 0$ same roots $r = \pm i\omega_0$

$(D^2+\omega_0^2)^2 y = 0$

$(r^2+\omega_0^2)^2 = 0 \to r = \pm i\omega_0$ mult: 2

$$y = (c_1 + c_3 t)\cos\omega_0 t + (c_2 + c_4 t)\sin\omega_0 t \quad \left(\begin{array}{c}\text{multiplicity 2} \\ \text{need linear function} \\ \text{coefficients}\end{array}\right)$$

$$= \underbrace{(c_1\cos\omega_0 t + c_2\sin\omega_0 t)}_{y_h \text{ (fixed amplitude)}} + \underbrace{t(c_3\cos\omega_0 t + c_4\sin\omega_0 t)}_{y_p \text{ (growing amplitude)}}$$

$\omega_0^2\,[y_p = t(c_3\cos\omega_0 t + c_4\sin\omega_0 t)]$

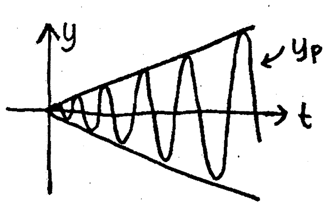
$0\,[Dy_p = t(-c_3\omega_0\sin\omega_0 t + c_4\omega\cos\omega_0 t) + (c_3\cos\omega_0 t + c_4\sin\omega_0 t)]$

$1\,[D^2y_p = t(-c_3\omega_0^2\cos\omega_0 t - c_4\omega_0^2\sin\omega_0 t) + 2(-c_3\omega_0\sin\omega_0 t + c_4\omega_0\cos\omega_0 t)]$

$$D^2y_p + \omega_0^2 y_p = t(0) + 2(-c_3\omega_0\sin\omega_0 t + c_4\omega_0\cos\omega_0 t) = B_0\cos\omega_0 t$$

$\therefore \left.\begin{array}{l} -2c_3\omega_0 = 0 \to c_3 = 0 \\ 2c_4\omega_0 = B_0 \to c_4 = B_0/(2\omega_0)\end{array}\right\} y_p = \left(\frac{B_0 t}{2\omega_0}\right)\sin\omega_0 t = A(t)\cos(\omega_0 t - \frac{\pi}{2})$ ← 90° out of phase [y_p in phase with $f(t)$]

$A(t) = \frac{B_0 t}{2\omega_0}$ linearly increasing amplitude



oscillation grows until model breaks down (no damping assumption no longer justified, etc)

HW Problem BEATING (5.6.1 extra)

$x'' + x = \cos(1.05t)$, $x(0) = 0$, $x'(0) = 0$

Note $\omega_0 = 1$, $\omega = 1.05 = 21/20$, $\frac{\omega_0 - \omega}{2} = \frac{-1}{40}$, $\frac{\omega_0 + \omega}{2} = \frac{41}{40}$.

The soln is proportional to a difference of cosines.
Use the identity $\cos A - \cos B = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$
to re-express the soln as a product of sines.
The beat period is $T_{beat} = \frac{2\pi}{\left|\frac{\omega_0 - \omega}{2}\right|}$, and the envelope of the soln is

given by the 2 curves: $x = \pm \text{Amplitude} \sin\left(\left|\frac{\omega_0-\omega}{2}\right| t\right)$

where "Amplitude" is the coefficient of the product of sines.

Plot the soln and envelope curves for one beat period, showing two beats of the soln oscillation.

Use the Maple template provided on-line to make this easier.