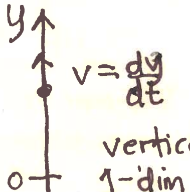
air resistance (separable DEs that don't involve the independent variable, again)

$$\left[\frac{dy}{dx} = f(y)\right.$$

f(y) linear or quadratic again, but now focusing on introducing dimensionless variables in an application]

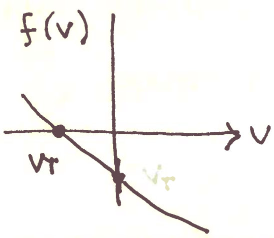


vertical 1-dim motion

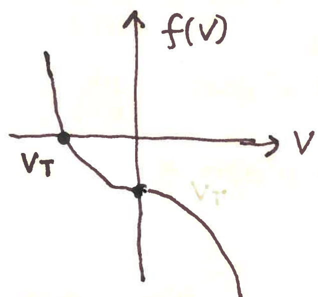
$$\underbrace{m\frac{dv}{dt}}_{\text{acceleration}} = \underbrace{-mg}_{\text{gravity down}} \overbrace{-k\,\mathrm{sign}(v)|v|^p}^{\text{resistance opposite to } v}, \quad k>0,\ 1\le p\le 2$$

$$\boxed{\frac{dv}{dt} = -g - \rho\,\mathrm{sign}(v)|v|^p}$$

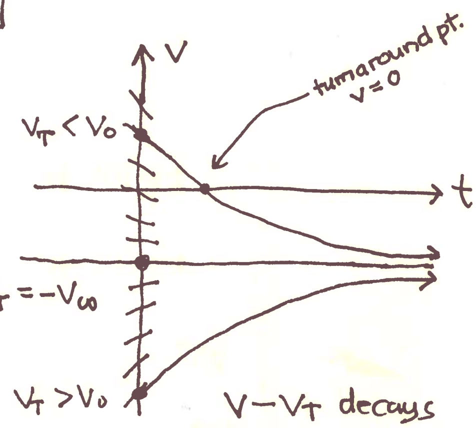
$\rho \equiv \frac{k}{m} > 0$ ("rho")



$p=1$: $f(v) = -g-\rho v$
linear



$p=2$: $f(v) = -g-\rho v|v|$
quadratic (piecewise)



$v - v_T$ decays away for all p

v_T = "terminal velocity"
v_∞ = "terminal speed"

dimensionless variables and characteristic constants

$$\frac{dv}{dt} = -g - \rho\,\mathrm{sign}(v)|v|^p = -g\left(1 + \frac{\mathrm{sign}(v)|v|^p}{(g/\rho)}\right)$$

$(g/\rho) \equiv v_\infty^p$ must have dimensions of $(\text{velocity})^p$

$$\boxed{v_\infty \equiv \left(\frac{g}{\rho}\right)^{1/p} > 0}$$ characteristic velocity → absolute value of terminal velocity

(sets scale on velocity axis)

$\frac{g}{\rho} = v_\infty^p \rightarrow g = \rho v_\infty^p$

$$= -\underbrace{\rho v_\infty^p}_{g}\left(1 + \mathrm{sign}(v)\frac{|v|^p}{v_\infty^p}\right)$$

$$= -g\left(1 + \mathrm{sign}(v)\left|\frac{v}{v_\infty}\right|^p\right)$$

$$\frac{1}{g}\frac{dv}{dt} = -\left[1 + \mathrm{sign}(v)\left|\frac{v}{v_\infty}\right|^p\right]$$ dimensionless r.h.s.

$$\underbrace{\frac{v_\infty}{g}}_{\equiv\tau}\frac{d(v/v_\infty)}{dt} = \frac{d(v/v_\infty)}{d(t/\tau)}$$

← dimensionless numerator
← denominator must be dimensionless

$$\boxed{\tau = \frac{v_\infty}{g} = \frac{1}{g}\left(\frac{g}{\rho}\right)^{1/p}}$$ characteristic time "tau"

(sets scale of tickmarks on time axis)

dimensionless variables:

$$\boxed{u = \frac{v}{v_\infty}, \quad T = \frac{t}{\tau}}$$

(measure in units of characteristic velocity and time)

$$\boxed{\frac{du}{dT} = -\left(1 + \mathrm{sign}(u)|u|^p\right)}$$

dimensionless D.E.

soln: $u = F(T)$

$\frac{v}{v_\infty} = F\left(\frac{t}{\tau}\right) \rightarrow v = v_\infty F\left(\frac{t}{\tau}\right)$

integrate to get position: $y = \int v\,dt$

air resistance (2)

$$\frac{du}{dT} = -\left(1+\mathrm{sign}(u)|u|^p\right)$$

$$\int \frac{du}{1+\mathrm{sign}(u)|u|^p} = -\int dT = -T + C_1$$

hyperbolic functions skipped in calc book

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

etc.
direct mirror of trigonometry — all formulas changed by signs

antiderivative formulas:

$p=1$: $\int \frac{du}{1+u} = \ln|u+1|$

$p=2$: $\mathrm{sign}(v)=1,\ v>0$: $\int \frac{du}{1+u^2} = \tan^{-1} u$

$\mathrm{sign}(v)=-1,\ v<0$: $\int \frac{du}{1-u^2} = \tanh^{-1} u = \frac{1}{2}\ln\frac{1+u}{1-u}$ if $|u|<1$

$= \coth^{-1} u = \frac{1}{2}\ln\frac{u-1}{u+1}$ if $|u|>1$

so solving for u:

$p=1$ $\quad |u+1| = \exp(C_1-T)$ or $u = -1 + C_2 e^{-T}$ $\rightarrow$ $\frac{v}{v_\infty} = -1 + C_2 e^{-t/\tau}$

$v = -v_\infty + C_3 e^{-t/\tau}$

$p=2,\ v>0$ $\quad u = \tan(C_1 - T)$

$v<0,\ |u|<1$: $\quad u = \tanh(C_1 - T)$

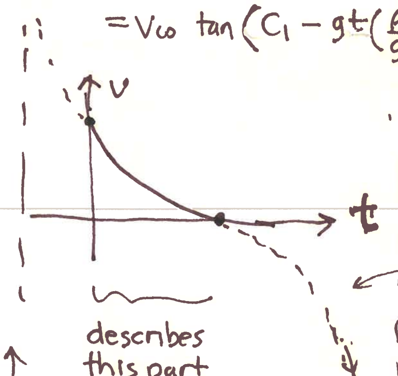
$|u|>1$: $\quad u = \coth(C_1 - T)$

$\rightarrow$ ditto.

just "details" of formulas, only possible for $p=1,\ p=2$ but not $1<p<2$.
need numerical solution.

for example:

$$v = v_\infty \tan(C_1 - t/\tau)$$
$$= v_\infty \tan\left(C_1 - gt\left(\frac{\varrho}{g}\right)^{1/p}\right) \qquad >0 \text{ case}$$



shape curve: "a tan(b−x)"

describes this part of $p=2$ solution.

tangent vertical asymptotes outside range of interest

EXAMPLE:

at $t=0,\ v=0,\ y=0$ $\quad$ drop from rest at zero height

$$\int_0^u \frac{dx}{1+\mathrm{sign}(x)|x|^p} = -T$$

$u=0$ when $T=0$

not very useful, just solve original DE directly then integrate:

$$y = \int_0^t v(t)\,dt$$

$y=0$ at $t=0$