

3 spring 2 mass system (no damping)

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) \quad x_1'' = -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1} (x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) + k_3 x_2 \quad x_2'' = -\frac{k_2}{m_2} (x_2 - x_1) + \frac{k_3}{m_2} x_2$$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \underline{x}'' = \underline{A} \underline{x}, \quad \underline{A} = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{(k_2+k_3)}{m_2} \end{bmatrix} = \begin{bmatrix} -(\omega_1^2 + \omega_2^2) & \omega_2^2 \\ \omega_2^2 & -(\omega_2^2 + \omega_3^2) \end{bmatrix}$

where $\gamma = \sqrt{\frac{m_2}{m_1}} > 0$ mass ratio
asymmetry parameter

$$\omega_1 = \sqrt{k_1/m_1} > 0 \text{ natural frequency of } (m_1, k_1) \text{ system}$$

$$\omega_2 = \sqrt{k_2/\sqrt{m_1 m_2}} > 0 \quad \gamma \omega_2 = \sqrt{k_2/m_1}, \quad \gamma^{-1} \omega_2 = \sqrt{k_2/m_2}$$

$$\omega_3 = \sqrt{k_3/m_2} > 0 \text{ natural frequency of } (m_2, k_3) \text{ system}$$

eigenvalues

$$0 = \det(\underline{A} - \lambda \underline{I}) = (\omega_1^2 + \gamma \omega_2^2 + \lambda)(\gamma^{-1} \omega_2^2 + \omega_3^2) - \omega_2^4 = \lambda^2 + [\omega_1^2 + (\gamma + \gamma^{-1}) \omega_2^2 + \omega_3^2] \lambda + [\omega_1^2 \omega_3^2 + \omega_2^2 (\gamma^{-1} \omega_1^2 + \gamma \omega_3^2)]$$

$$\lambda = \lambda_{\pm} = \frac{1}{2} \left\{ -[\omega_1^2 + (\gamma + \gamma^{-1}) \omega_2^2 + \omega_3^2] \pm \sqrt{[\omega_1^2 + (\gamma + \gamma^{-1}) \omega_2^2 + \omega_3^2]^2 - 4[\omega_1^2 \omega_3^2 + \omega_2^2 (\gamma^{-1} \omega_1^2 + \gamma \omega_3^2)]} \right\}$$

maple needed to find these relations

$$\left\{ \begin{array}{l} b \\ b^2 - D = 4 \omega_2^2 (\gamma^{-1} \omega_1^2 + \omega_1^2 \omega_3^2 + \gamma \omega_3^2) > 0 \end{array} \right. \quad \text{so } b > D^{1/2}, \quad -b + D^{1/2} < 0$$

conclusion $\lambda_+ < 0$, obviously $\lambda_- < 0$; $\lambda_+ - \lambda_- = \frac{(-b + D^{1/2}) - (-b - D^{1/2})}{2} = D^{1/2} > 0$

so $|\lambda_{\pm}|$ both negative, λ_+ is greater than λ_- : $\lambda_- < \lambda_+ < 0 \quad \therefore |\lambda_+| > |\lambda_-|$

But we assumed $D \geq 0$ for real eigenvalues.

$$D = \underbrace{X^2}_{A=1} + \underbrace{2(\gamma - \gamma^{-1})XY}_{B} + \underbrace{\gamma^2 + \gamma^{-2}}_{C} Y^2 \quad \text{where } X = \omega_1^2 - \omega_3^2, \quad Y = \omega_2^2$$

$$\text{is a quadratic function of } X, Y; \text{ its discriminant is: } \left\{ \begin{array}{l} \text{Disc} = B^2 - 4AC = 4(\gamma - \gamma^{-1})^2 - 4(\gamma^2 + \gamma^{-2})^2 \\ = 4[\gamma^2 - 2 + \gamma^{-2} - (\gamma^2 + 2 + \gamma^{-2})] = -16 < 0 \end{array} \right.$$

This guarantees $D > 0$. If we try to solve $D = 0$ along any line $X = mY$ we get

$(Am^2 + Bm + C)Y^2 = 0$ but the quad eq has only complex roots so it is never zero. Since $X = Y = 1 \rightarrow D = 3\gamma^2 > 0$, it cannot change sign.