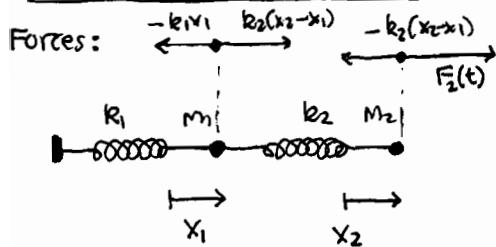


2 mass 2 spring system



$$\begin{aligned} m_1 \ddot{x}_1'' &= -k_1 x_1 + k_2 (x_2 - x_1) = -(k_1 + k_2) x_1 + k_2 x_2 \\ m_2 \ddot{x}_2'' &= -k_2 (x_2 - x_1) + F_2 = k_2 x_1 - k_2 x_2 + F_2 \\ \ddot{x}_1'' &= -\frac{(k_1 + k_2)}{m_1} x_1 + \frac{k_2}{m_1} x_2 \quad (\text{special case where } F_2 = 0) \\ \ddot{x}_2'' &= \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_2 + \frac{F_2}{m_2} \end{aligned}$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}'' = \begin{bmatrix} -(k_1 + k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -k_2/m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \quad (\text{standard form: } \ddot{\underline{x}}'' = A \underline{x})$$

gives free motion

$$A = \begin{bmatrix} -\frac{(k_1 + k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{bmatrix}$$

driving term for second mass only

periodic motion example: $m_1 = 2$
 $m_2 = 1$

$$k_1 = 9 \quad k_2 = 2 \rightarrow A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{cases} \lambda = -1, -4 \\ B = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} = [b_1 \ b_2] \\ B^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix}, A_B = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \end{cases}$$

decoupled equations: $B^{-1} \underline{f} = \frac{1}{3} \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \cos 3t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \cos 3t \\ \frac{1}{3} \cos 3t \end{bmatrix}$

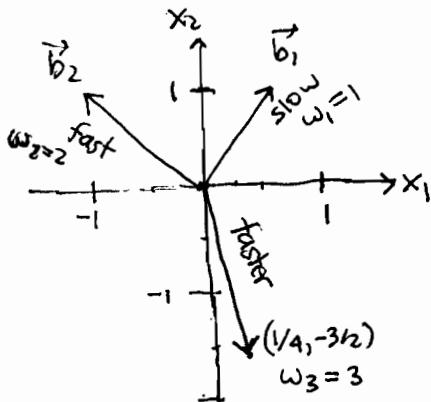
$$\begin{aligned} \ddot{\underline{x}} &= A \underline{x} + B^{-1} \underline{f} \rightarrow \begin{aligned} \ddot{y}_1'' &= -y_1 + \frac{2}{3} \cos 3t \\ \ddot{y}_2'' &= -4y_2 + \frac{1}{3} \cos 3t \end{aligned} \rightarrow \begin{aligned} \ddot{y}_1'' + y_1 &= \frac{2}{3} \cos 3t \rightarrow y_{1h} = c_1 \cos t + c_2 \sin t \\ \ddot{y}_2'' + 4y_2 &= \frac{1}{3} \cos 3t \rightarrow y_{2h} = c_3 \cos 2t + c_4 \sin 2t \end{aligned} \end{aligned}$$

$$y_{1p} = c_5 \cos 3t: \quad y_{1p}'' + y_{1p} = (-9c_5 \cos 3t) + (c_5 \cos 3t) = -8c_5 \cos 3t = \frac{2}{3} \cos 3t \rightarrow c_5 = -\frac{2}{3 \cdot 8} = -\frac{1}{12}$$

$$y_{2p} = c_6 \cos 3t: \quad y_{2p}'' + 4y_{2p} = (-9c_6 \cos 3t) + 4(c_6 \cos 3t) = -5c_6 \cos 3t = \frac{1}{3} \cos 3t \rightarrow c_6 = -\frac{1}{15}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t & -\frac{1}{12} \cos 3t \\ c_3 \cos 2t + c_4 \sin 2t & -\frac{1}{15} \cos 3t \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos t + c_2 \sin t - \frac{1}{12} \cos 3t \\ c_3 \cos 2t + c_4 \sin 2t - \frac{1}{15} \cos 3t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}(c_1 \cos t + c_2 \sin t) - (c_3 \cos 2t + c_4 \sin 2t) + f \left(\frac{1}{24} + \frac{1}{15} \right) \cos 3t \\ (c_1 \cos t + c_2 \sin t) + (c_3 \cos 2t + c_4 \sin 2t) + \left(-\frac{1}{12} - \frac{1}{15} \right) \cos 3t \end{bmatrix}}_{3/20}$$



$$\begin{aligned} \ddot{x}_h &= ((\cos t + c_2 \sin t) \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + (c_3 \cos 2t + c_4 \sin 2t) \begin{bmatrix} -1 \\ 1 \end{bmatrix}) \\ &= \frac{\cos 3t}{10} \begin{bmatrix} 1/4 \\ -3/2 \end{bmatrix} \end{aligned}$$

"eigenmodes" of free motion – frequencies 1 and 2
"eigenfrequencies"
along eigendirections

initial conditions determine free motion part of soln.

$$\omega_1 = \sqrt{-\lambda_1} = 1, \quad \omega_2 = \sqrt{-\lambda_2} = 2$$

$$T_1 = 2\pi, \quad T_2 = \frac{2\pi}{2} = \pi$$

$$\omega_3 = 3$$

$$T_3 = \frac{2\pi}{3}$$

since ratios $T_1 : T_2 : T_3 \sim 3 : 2 : 1$ are integers

smallest common period is 2π during which:

- mode 1: 1 oscillation
- mode 2: 2 oscillations
- mode 3: 3 oscillations

motion of system is periodic with period 2π

initial conditions at rest at equilibrium

$$\underline{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \dot{\underline{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

leads to

$$\underline{x} = \frac{1}{12} \cos t \vec{b}_1 + \frac{1}{15} \cos 2t \vec{b}_2 + \frac{1}{10} \cos 3t \begin{bmatrix} 1/4 \\ -3/2 \end{bmatrix}$$